A Schur Thing -
Appendix to As Above, So Below: The Up-Down Operators for the (m)-Associahedra Partition Polynomials

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The goal of this appendix is to show that the ladder operators under substitution of indeterminates for the sets $[A^{(m)}]$ of the (m)-associahedra partition polynomials $A_n^{(m)}(u_1, ..., u_n)$ are the set $[N^{(1)}] = [N]$ of noncrossing partition polynomials $N_n^{(1)}(u_1, ..., u_n) = N_n(u_1, ..., u_n)$ and its inverse $[N^{(-1)}] = [N]^{-1}$ of the partition polynomials $N_n^{(-1)}(u_1, ..., u_n)$. More precisely, $[N]$ is the left-sided raising op and the right-sided lowering op, and $[N]^{-1}$ is the left-sided lowering op and the right-sided raising op; that is, (borrowing terms from quantum physics) the up / raising / creation operations are

$$[N][A^{(m)}] = [A^{(m)}][N]^{-1} = [A^{(m+1)}],$$

and the down / lowering / annihilation / destruction operations are

$$[N]^{-1}[A^{(m)}] = [A^{(m)}][N] = [A^{(m-1)}].$$

It then follows, recalling $[A^{(0)}] = [R]$, from the defining relations

$$[A^{(m)}] = [N^{(m)}][A^{(0)}] = [N^{(m)}][R]$$

that the set $[N^{(m)}]$ of (m)-noncrossing partitions polynomials $N_n^{(m)}(u_1, ..., u_n)$ are generated via iterated substitution as

$$[N^{(m)}] = [N]^m.$$

In this appendix, compositional inversion of a function is flagged with the superscript $< -1 >$ whereas the superscript (-1) designates only sets of partition polynomials.

The Lagrange-Schur-Jabotinsky self-convolution formula

As presented in Section I of “As Above, So Below”, the basic identity

$$[A] = [N][R]$$

relates the set $[A^{(1)}] = [A]$ of fundamental associahedra partition polynomials for direct compositional inversion of power series / ordinary generating functions (o.g.f.s) to the set $[N^{(1)}] = [N]$ noncrossing partition polynomials for indirect compositional inversion of power series via the set $[A^{(0)}] = [R]$ of reciprocal partition polynomials. This is reflected in the classic Lagrange inversion formula (LIF) for generating the power series coefficients of the compositional inverse $F^{<-1>}(x)$ of an invertible analytic function $F(x)$, for $n \geq 0$,

$$[x^{n+1}] \; F^{<-1>}(x) = [x^n] \; \frac{F^{<-1>}(x)}{x} = [x^n] \frac{\left(\frac{x}{F(x)}\right)^{n+1}}{n+1}$$

$$= \frac{D^{n+1}_{x=0}}{(n+1)!} \; F^{<-1>}(x) = \frac{D^{n}_{x=0}}{n!} \; \frac{F^{<-1>}(x)}{x} = \frac{D^{n}_{x=0}}{n!} \; \frac{\left(\frac{x}{F(x)}\right)^{n+1}}{n+1}$$

for

$$F(x) = c_1 x + c_2 x^2 + c_3 x^3 + \cdots \; .$$

The LIF is a specialization of the Lagrange-Schur-Jabotinsky self-convolution formula (LSJF)

$$[x^n] \frac{\left(\frac{F^{<-1>}(x)}{x}\right)^{k}}{k} = [x^n] \frac{\left(\frac{x}{F(x)}\right)^{n+k}}{n+k},$$

and it's a sure bet that the more general identity

$$[A^{(m+1)}] = [N][A^{(m)}]$$

issues from the LSJF. This will be shown below.

The LSJ identity for n, k positive integers is essentially eqn. 65 on p. 25 of "Identities in the theory of power series" by Schur, published posthumously in American Journal of Mathematics, Volume 69, No. 1, Jan 1947. Jabotinsky subsequently extended this to all integers (see the discussion in "Lagrange inversion" by Gessel).

Stanley on pg. 38 of Enumerative Combinatorics II in Thm. 5.4.2 has for k, n any integers

$$[x^n]\frac{(F^{<-1>}(x))^k}{k} = [x^{n-k}]\frac{(\frac{x}{F(x)})^n}{n}.$$

Then

$$[x^n]x^k\frac{(\frac{F^{<-1>}(x)}{x})^k}{k} = [x^{n-k}]\frac{(\frac{F^{<-1>}(x)}{x})^k}{k} = [x^{n-k}]\frac{(\frac{x}{F(x)})^n}{n},$$

and, with $n \to n+k$,

$$[x^n]\frac{(\frac{F^{<-1>}(x)}{x})^k}{k} = [x^n]\frac{(\frac{x}{F(x)})^{n+k}}{n+k},$$

the Schur identity, but now valid for n, k any integers (with limiting procedures applied when a denominator vanishes).

For $k = 1$, the Schur identity gives

$$[x^n]\ \frac{F^{<-1>}(x)}{x} = [x^{n+1}]\ F^{<-1>}(x) = [x^n]\frac{(\frac{x}{F(x)})^{n+1}}{n+1},$$

$$= \frac{D^{n+1}_{x=0}}{(n+1)!}F^{<-1>}(x) = \frac{D^n_{x=0}}{n!}\frac{(\frac{x}{F(x)})^{n+1}}{n+1},$$

the classic LIF for $n \geq 0$, as noted above, one avenue for establishing that $[A] = [N][R]$.

For $k = -1$, the Schur identity gives

$$[x^n]\ \frac{x}{F^{<-1>}(x)} = [x^n]\frac{-(\frac{x}{F(x)})^{n-1}}{n-1}$$

$$= \frac{D^n_{x=0}}{n!}\ \frac{x}{F^{<-1>}(x)} = \frac{D^n_{x=0}}{n!}\frac{-(\frac{x}{F(x)})^{n-1}}{n-1}$$

$$= [x^{n-1}]\frac{1}{F^{<-1>}(x)} = [x^n]\frac{\left(\frac{F(x)}{x}\right)^{-n+1}}{-n+1} = \frac{D^n_{x=0}}{n!}\frac{\left(\frac{F(x)}{x}\right)^{-n+1}}{-n+1}.$$

This determines the coefficients

$$N_n^{(-1)}(u_1, ..., u_n) = [x^n]\ \frac{x}{F^{<-1>}(x)}$$

of $N^{(-1)}(x)$ for $n \geq 0$ with

$$\frac{1}{A(x)} = \frac{1}{1 + A_1(u_1)x + A_2(u_1, u_2)x^2 + \cdots} = N^{(-1)}(x)$$

$$= 1 + N_1^{(-1)}(u_1)x + N_2^{(-1)}(u_1, u_2)x^2 + \cdots,$$

or, equivalently,

$$[R][A] = [N]^{-1},$$

which follows from the inverse of $[N] = [A][R] = [A][A^{(0)}]$. Then $[R] = [A^{(0)}] = [N]^{-1}[A]$, recalling $[A^{(m)}]^{-1} = [A^{(m)}]$, suggesting the Schur identity might be a means via which to prove the more general lowering relation $[A^{(-m-1)}] = [N]^{-1}[A^{(-m)}]$, implying $[A^{(-m-1)}] = [N]^{-m-1}[A^{(0)}] = [N]^{-m-1}[R]$, for $m \geq 0$.

$[N]$ as the left-sided raising op for $[A^{(m)}]$ for $m \geq 0$.

Once $[N]$ is established to satisfy $[N][A^{(m)}] = [A^{(m+1)}]$ for $m \geq 0$, the validity of the left-sided lowering operation $[A^{(m)}] = [N]^{-1}[A^{(m+1)}] = [A^{(m+1)}][N]$ for $m \geq 0$ follows.

We already have from the literature on the relation between compositional inversion and noncrossing partitions that $[A] = [A^{(1)}] = [N^{(1)}][A^{(0)}] = [N][R]$. I first came across the geometric half of this in “McCammond, Non-crossing Partitions in Surprising Locations” by McCammond after first generating via my own approach the partition polynomials $[N]$ as coefficients for compositional inversion of an o.g.f. in terms of its shifted reciprocal. Using the

LSJ identity, I'll first show that $[N][A^{(1)}] = [A^{(2)}]$ and then that $[N][A^{(2)}] = [A^{(3)}]$. The proof of the general identity $[N][A^{(m)}] = [A^{(m+1)}]$ for $m > 0$ follows from an obvious generalization of this line of arguments.

Reprising some notation, the compositional inverse of the o.g.f.

$$O(x;m) = x(1 + u_1 x^m + u_2 x^{2m} + u_3 x^{3m} + \cdots) = \frac{O(x^m;1)}{x^{m-1}} = \frac{O(x^m)}{x^{m-1}}$$

for $m > 0$ is

$$(O(x;m))^{<-1>} = O^{<-1>}(x;m)) = xA^{(m)}(x^m)$$

$$= x(1 + A_1^{(m)}(u_1)x^m + A_2^{(m)}(u_1,u_2)x^{2m} + \cdots) = xN^{(m)}(x^m)$$

$$= x(1 + N_1^{(m)}(R_1(u_1))x^m + N_2^{(m)}(R_1(u_1), R_2(u_1,u_2)x^{2m} + \cdots),$$

or, equivalently in terms of the partition polynomials only,

$$[A^{(m)}][u] = [N^{(m)}][R][u] = [N^{(m)}][A^{(0)}][u],$$

or, removing the dummy indeterminates / extracting the operators,

$$[A^{(m)}] = [N^{(m)}][R] = [N^{(m)}][A^{(0)}].$$

For $m = 2$, I wish to prove

$$[N][A^{(1)}] = [N][A] = [A][R][A] = [A^{(2)}].$$

The last equality is equivalent to

$$(\frac{x^2}{xA(x)})^{<-1>} = (xN^{(-1)}(x))^{<-1>} = xA^{(2)}(x),$$

which in turn is equivalent to

$$(\frac{x^2}{O^{<-1>}(x)})^{<-1>}|_{x \to x^2} = x\, O^{<-1>}(x;2) = x\,(\frac{O(x^2)}{x})^{<-1>}.$$

So, with

$$(\frac{O(x^2)}{x})^{<-1>} = x + c_1^{(2)}x^3 + c_2^{(2)}x^5 + \cdots = xA^{(2)}(x^2)$$

and

$$(\frac{x^2}{O^{<-1>}(x)})^{<-1>} = x + d_1x^2 + d_2x^3 + \cdots ,$$

I need to prove

$$c_n^{(2)} = [x^{2n+1}](\frac{O(x^2)}{x})^{<-1>} = [x^n](\frac{x^2}{O^{<-1>}(x)})^{<-1>} = d_n.$$

From the LIF, with $z = x^2$,

$$c_n^{(2)} = [x^{2n+1}](\frac{O(x^2)}{x})^{<-1>} = [x^{2n}]\frac{(\frac{x^2}{O(x^2)})^{2n+1}}{2n+1}$$

$$= [z^n]\frac{(\frac{z}{O(z)})^{2n+1}}{2n+1} = [x^n]\frac{(\frac{x}{O(x)})^{2n+1}}{2n+1},$$

and, from applying the Schur identity with $k = n + 1$ and then the LIF,

$$c_n^{(2)} = [x^n]\frac{(\frac{x}{O(x)})^{2n+1}}{2n+1} = [x^n]\frac{(\frac{O^{<-1>}(x)}{x})^{n+1}}{n+1} = [x^{n+1}](\frac{x^2}{O^{<-1>}(x)})^{<-1>} = d_n$$

with $c_0 = d_0 = 1$, which proves that

$$c_n^{(2)} = d_n$$

and, consequently,

$$[N][A^{(1)}] = [N][A] = [A][R][A] = [A^{(2)}].$$

Since $[A^{(m)}] = [A^{(m)}]^{-1}$ and $[A^{(2)}] = [N^{(2)}][A^{(0)}] = [N][A] = [N][N][A^{(0)}] = [N]^2[A^{(0)}]$,

then

$$[A^{(2)}] = [A^{(2)}]^{-1} = ([N]^2[A^{(0)}])^{-1} = [A^{(0)}][N]^{-2} = [A][N]^{-1}$$

and

$$[N^{(2)}] = [N]^2.$$

For $m = 3$, I wish to prove

$$[N][A^{(2)}] = [A][R][A^{(2)}] = [A^{(3)}].$$

The last equality is equivalent to

$(\dagger)$

$$(\frac{x^2}{xA^{(2)}(x)})^{<-1>} = xA^{(3)}(x).$$

Since

$$A^{(m)}(x) = 1 + A_1^{(m)}(u_1)x + A_2^{(m)}(u_1, u_2)x^2 + \cdots ,$$

and, for $m > 0$,

$$O^{<-1>}(x; m) = (\frac{O(x^m)}{x^{m-1}})^{<-1>} = xA^{(m)}(x^m)$$

$$= x(1 + A_1^{(m)}(u_1)x^m + A_2^{(m)}(u_1, u_2)x^{2m} + A_3^{(m)}(u_1, u_2, u_3)x^{3m} + \cdots),$$

$(\dagger)$ is equivalent to

$$(\frac{x^2}{xA^{(2)}(x)})^{<-1>}|_{x \to x^3} = x^2\ O^{<-1>}(x; 3) = x^2\ (\frac{O(x^3)}{x^2})^{<-1>}.$$

This together with the LIF implies

$(\dagger\dagger)$

$$[x^n]\frac{(A^{(2)}(x))^{n+1}}{n+1} = [x^{3n}]\frac{(\frac{x^3}{O(x^3)})^{3n+1}}{3n+1}.$$

The LHS of this (assumed) coefficient equality $(\dagger\dagger)$ can be expressed as

$$[x^n]\frac{(A^{(2)}(x))^{n+1}}{n+1} = [x^{2n}]\frac{(\frac{O^{<-1>}(x;2)}{x})^{n+1}}{n+1},$$

and the RHS, as

$$[x^{3n}]\frac{(\frac{x^3}{O(x^3)})^{3n+1}}{3n+1} = [x^n]\frac{(\frac{x}{O(x)})^{3n+1}}{3n+1} = [x^n]\frac{(h(x))^{3n+1}}{3n+1}.$$

Furthermore, the Schur identity with $n \to 2n$ and $k \to n+1$ gives for the LHS

$$[x^n]\frac{(A^{(2)}(x))^{n+1}}{n+1} = [x^{2n}]\frac{(\frac{O^{<-1>}(x;2)}{x})^{n+1}}{n+1} = [x^{2n}]\frac{(\frac{x}{O(x;2)})^{3n+1}}{3n+1}$$

$$= [x^{2n}]\frac{(\frac{x^2}{O(x^2)})^{3n+1}}{3n+1} = [x^n]\frac{(\frac{x}{O(x)})^{3n+1}}{3n+1}$$

and this last expression is one form of the RHS, so the identities $(\dagger\dagger)$ and $(\dagger)$ are indeed true.

For $m \geq 0$, the same arguments apply with 2 replaced by m and 3, by $m+1$, so we can conclude that for $m \geq 0$

$$[A^{(m+1)}] = [N][A^{(m)}] = [N^{(m)}][A^{(0)}] = [N^{(m)}][R] = [N]^m[A^{(0)}] = [N]^m[R],$$

and, taking inverses,

$$[A^{(m+1)}] = [A^{(m)}][N]^{-1} = [A^{(0)}][N^{(m)}]^{-1} = [R][N^{(m)}]^{-1}$$

$$= [A^{(0)}][N]^{-m} = [R][N]^{-m}.$$

In addition, since

$$[A^{(m+1)}] = [N][A^{(m)}] = [A][R][A^{(m)}],$$

other reps of these results are

$$(\frac{x^2}{xA^{(m)}(x)})^{<-1>} = xA^{(m+1)}(x)$$

and

$$(\frac{x^2}{xA^{(m)}(x)})^{<-1>}|_{x\to x^{m+1}} = x^m O^{<-1>}(x;m+1).$$

In other of my notes, I've identified $xA^{(m)}(x) = O^{<-1>}(x;m+1)_{dea}$ as the deaerated version of $O^{<-1>}(x;m+1)$ and given the formulas

$$(\frac{x^2}{O^{<-1>}(x;m)_{dea}})^{<-1>} = O^{<-1>}(x;m+1)_{dea}$$

or

$$(\frac{x^2}{O^{<-1>}(x;m)_{dea}})^{<-1>}|_{x\to x^{m+1}} = x^m O^{<-1>}(x;m+1).$$

The lowering operation for $[A^{(m)}]$ for $m \leq 0$.

The raising and lowering operations for $[A^{(m)}]$ for $m \geq 0$ have been characterized above. Now let's look at the lowering operation for $[A^{(m)}]$ for $m \leq 0$.

For $k = -1$, the Schur identity gives

(††)

$$[x^n]\left(\frac{x}{F^{<-1>}(x)}\right) = [x^n]\frac{-\left(\frac{x}{F(x)}\right)^{n-1}}{n-1} = [x^n]\frac{\left(\frac{F(x)}{x}\right)^{-n+1}}{-n+1}.$$

The first local, coefficient expression is equivalent to the global expression

$[R][A][F] = [N]^{-1}[F],$

or, abstracting,

$[R][A] = [N]^{-1}$

and

$$\frac{x^2}{xA(x)} = xN^{(-1)}(x)$$,

or

$$\frac{1}{A(x)} = N^{(-1)}(x)$$.

Then the conjugation

$$[R][A][R] = [N]^{-1}[R] = [N^{-1}][R] = [A^{(-1)}]$$

holds with the last two equalities established in my earlier notes, e.g., "One Matrix to Rule Them All", more or less as the definition of the set of special Schur self-convolution expansion coefficients, which I have often denoted by $[b]$ or $[K]$ in other notes.

In this section of the appendix, I wish to prove for $m \leq 0$ the lowering operation

$$[A^{(m-1)}] = [N]^{-1}[A^{(m)}] = [R][A][A^{(m)}],$$

or for $p \geq 0$

$$[N^{(-p-1)}][R] = [A^{(-p-1)}] = [N]^{-1}[A^{(-p)}] = [R][A][A^{(-p)}] = [N]^{-1}[N^{(-p)}][R].$$

Equivalently,

$$A^{(-p-1)}(x) = \frac{x}{(xA^{(-p)}(x))^{<-1>}}.$$

Note that with the relabeling $|-p| = q$, $[M] = [N^{(-1)}] = [N]^{-1}$, and $[A^{(-p)}] = [B^{(q)}]$, the lowering operation can be recast as the raising operation for $q \geq 0$

$$[B^{(q+1)}] = [M][B^{(q)}] = [M]^q[B^{(0)}] = [M]^q[R].$$

Returning to the main argument, with

$$F(x) = x(1 + u_1x + u_2x^2 + u_3x^3 + u_4x^4 + \cdots)$$

and

$$G(x)=\frac{x^2}{F(x)}$$

so that $[G]=[R][F]$,

the Schur identity gives

(String 1)

$$[x^n]\left(\frac{x}{G^{<-1>}(x)}\right)=[x^n]\frac{\left(\frac{G(x)}{x}\right)^{(-n+1)}}{-n+1}$$

$$=[x^n]\left(\frac{x}{(\frac{x^2}{F(x)})^{<-1>}}\right)=[x^n]\frac{\left(\frac{F(x)}{x}\right)^{-(-n+1)}}{-n+1},$$

or, equivalently, since $[G]=[R][F]$ and $[N^{(-1)}]=[N]^{-1}=[A^{(-1)}][R]$,

$$[R][A][G]=[N]^{-1}[G]$$

$$=[R][A][R][F]=[N]^{-1}[R][F]=[A^{(-1)}][F],$$

implying,

$$[A^{(-1)}]=[N]^{-1}[R]=[R][A][R]=[R][N],$$

or

$$A^{(-1)}(x)=\frac{1}{N(x)}.$$

Since $[R]^2=[A^{(0)}]^2=[I]$, these imply the conjugation relations

$$[A^{(-1)}]=[R][A][R]$$

and

$[N]^{-1} = [R][N][R].$

With the specialization $F(x) = O(x^2)/x = O(x,2)$,

$$[x^n]\frac{\left(\frac{F(x)}{x}\right)^{-(-n+1)}}{-n+1} = [x^n]\left(\frac{x}{(\frac{x^2}{F(x)})^{<-1>}}\right)$$

in (String 1) becomes

$$[x^n]\frac{\left(\frac{O(x^2)}{x^2}\right)^{-(-n+1)}}{-n+1} = [x^n]\left(\frac{x}{(\frac{x^3}{O(x^2)})^{<-1>}}\right),$$

giving the non vanishing components

$$[x^{2n}]\frac{\left(\frac{O(x^2)}{x^2}\right)^{-(-2n+1)}}{-2n+1} = [x^{2n}]\left(\frac{x}{(\frac{x^3}{O(x^2)})^{<-1>}}\right).$$

From the relation of $[A^{(-|m|)}]$ to $[A^{(-1)}]$ given in Section 1 of "As Above, So Below",

$$A_n^{(-2)} = [x^{2n}]\frac{\left(\frac{O(x^2)}{x^2}\right)^{-(-2n+1)}}{-2n+1} = [x^n]\frac{\left(\frac{O(x)}{x}\right)^{-(-2n+1)}}{-2n+1},$$

so

$$A_n^{(-2)} = [x^{2n}]\left(\frac{x}{(\frac{x^3}{O(x^2)})^{<-1>}}\right).$$

Now

$$\frac{x}{\frac{x^3}{O(x^2)}} = \frac{O(x^2)}{x^2},$$

so, by the LIF, the nonzero coefficients of $(\frac{x^3}{O(x^2)})^{<-1>}$ are

$$[x^{2n}]\frac{(\frac{O(x^2)}{x^2})^{2n+1}}{2n+1} = [x^n]\frac{(\frac{O(x)}{x})^{2n+1}}{2n+1} = N_n^{(2)}.$$

Then

$$A^{(-2)}(x) = \frac{1}{N^{(2)}(x)},$$

or

$$[A^{(-2)}] = [R][N^{(2)}] = [R][N]^2.$$

Consequently,

$$[N^{(-2)}][R] = [A^{(-2)}] = [A^{(-2)}]^{-1} = ([R][N^{(2)}])^{-1} = ([R][N]^2)^{-1} = [N]^{-2}[R],$$

so

$$[N^{(-2)}] = [N]^{-2}.$$

This same line of argument can be generalized to $[A^{(m)}]$ for $m < -2$ by letting $F(x) = \frac{O(x^{|m|})}{x^{|m|-1}} = O(x;m)$, i.e., by replacing 2 by m and 3 by $m+1$, to verify

$$[A^{(-|m|)}] = [N^{(-|m|)}][R] = [N]^{-|m|}[R]$$

and, therefore,

$$[N^{(-|m|)}] = [N]^{-|m|},$$

$$[A^{(-|m+1|)}] = [N]^{-1}[A^{(-|m|)}],$$

$$[A^{(-|m+1|)}] = [A^{(-|m|)}][N],$$

and

$$[A^{(-|m|)}] = [N][A^{(-|m|)}][N].$$

This last is easily checked to be valid for $m = 0$ by

$[A^{(-|0|)}] = [R]$

and

$[N][A^{(-|0|)}][N] = [N][R][N] = [A][R][R][A][R] = [A][A][R] = [R].$

Reprising

I've shown the validity, for m any integer, of the raising operations

$[N][A^{(m)}] = [A^{(m)}][N]^{-1} = [A^{(m+1)}]$

and the lowering operations

$[N]^{-1}[A^{(m)}] = [A^{(m)}][N] = [A^{(m-1)}].$

These imply

$[N]^m[A^{(0)}] = [N]^m[R] = [A^{(m)}]$,

which implies with

$[N^{(m)}][R] = [A^{(m)}]$

that

$[N^{(m)}] = [N]^m.$

Spot Checks / Illustrations:

I)

Checking the $(\dagger\dagger)$ identity:

For $k = -1$, the Schur identity gives

$$[x^n]\ \frac{x}{F^{<-1>}(x)} = [x^n]\frac{-(\frac{x}{F(x)})^{n-1}}{n-1}$$

$$= \frac{D^n_{x=0}}{n!}\ \frac{x}{F^{<-1>}(x)} = \frac{D^n_{x=0}}{n!}\frac{-(\frac{x}{F(x)})^{n-1}}{n-1}$$

$$= [x^{n-1}]\frac{1}{F^{<-1>}(x)} = [x^n]\frac{(\frac{F(x)}{x})^{-n+1}}{-n+1}$$

$$= \frac{D^n_{x=0}}{n!}\frac{(\frac{F(x)}{x})^{-n+1}}{-n+1}.$$

Using Wolfram Alpha (WA) Online with $F(x) = x(1 + u_1 x + u_2 x^2 + \cdots)$ to check the last expression,

zeroth derivative (1/0!) (1 +u_1 x + u_2x^2 + u_3x^3+ u_4 x^4 + u_5x^5)^(-0+1) / (-0+1) at x=0 is $1 = N_0^{(-1)}(u_1)$,

first derivative (1/1!) (1 +u_1 x + u_2x^2 + u_3x^3+ u_4 x^4 + u_5x^5)^(-1+1.0001) / (-1+1.0001) at x=0 is $u_1 = N_1^{(-1)}(u_1)$,

second derivative (1/2!) (1 +u_1 x + u_2x^2 + u_3x^3+ u_4 x^4 + u_5x^5)^(-2+1) / (-2+1) at x=0 is $u_2 - u_1^2 = N_2^{(-1)}(u_1, u_2)$,

third derivative (1/3!) (1 +u_1 x + u_2x^2 + u_3x^3+ u_4 x^4 + u_5x^5)^(-3+1) / (-3+1) at x=0 is $2u_1^3 - 3u_2 u_1 + u_3 = N_3^{(-1)}(u_1, u_2, u_3)$,

and

fourth derivative (1/4!) (1 +u_1 x + u_2x^2 + u_3x^3+ u_4 x^4 + u_5x^5)^(-4+1) / (-4+1) at x = 0 is $-5u_1^4 + 10u_2 u_1^2 - 4u_3 u_1 - 2u_2^2 + u_4 = N_4^{(-1)}(u_1, ..., u_4)$.

The above is in agreement with https://oeis.org/A350499 for $[N]^{-1} = [N^{(-1)}]$.

Using the first few partition polynomials of $[A]$ to check the first expression,

$$\frac{x}{F^{<-1>}(x)} = \frac{x}{xA(x)} = \frac{1}{1 + A_1(u_1)x + A_2(u_1,u_2)x^2 + \cdots}$$

$$= \frac{1}{1 + (-u_1)x + (2u_1^2 - u_2)x^2 + (-5u_1^3 + 5u_1u_2 - u_3)x^3 + (14u_1^4 - 21u_1^2u_2 + 6u_1u_3 + 3u_2^2 - u_4)x^4 + \cdots}$$

and WA gives

series 1 / (1 + (- u_1) x + (2 u_1^2 - u_2) x^2 + (- 5 u_1^3 + 5 u_1 u_2 - u_3) x^3 + (14 u_1^4 - 21 u_1^2 u_2 + 6 u_1 u_3 + 3 u_2^2 - u_4) x^4)

as

1 + u_1 x + (u_2 - u_1^2) x^2 + (2 u_1^3 - 3 u_2 u_1 + u_3) x^3 + (-5 u_1^4 + 10 u_2 u_1^2 - 4 u_3 u_1 - 2 u_2^2 + u_4) x^4 + ...

II)

Checking the $\dagger$ identity:

$$(\frac{x^2}{xA^{(2)}(x)})^{<-1>} = xA^{(3)}(x).$$

Using the LIF, for

$$\frac{D^3_{x=0}}{3!}(A^{(2)}(x))^{3+1}/(3+1)$$

WA gives

third derivative (1/3!) (1 + (-u_1)x +(- u_2 + 3 u_1^2)x^2 + (- u_3 + 8 u_1 u_2 - 12 u_1^3)x^3)^(3+1) / (3+1) at x=0 is $-22u_1^3 + 11u_2u_1 - u_3 = A_3^{(3)}$,

and, for direct computation of the coefficient $A_3^{(3)}$ using the LIF,

$$A_3^{(3)} = [x^{3\cdot 3}]\frac{\left(\frac{x^3}{O(x^3)}\right)^{3\cdot 3+1}}{3\cdot 3+1} = [x^3]\frac{\left(\frac{x}{O(x)}\right)^{3\cdot 3+1}}{3\cdot 3+1},$$

WA gives

ninth derivative (1/9!) (1 / (1 + u_1x^3 + u_2 x^6 + u_3x^9))^(3\cdot 3+1) / (3\cdot 3+1) at x=0 is $-22u_1^3 + 11u_1u_2 - u_3 = A_3^{(3)}$

and

third derivative (1/3!) (1 / (1 + u_1x + u_2 x^2 + u_3x^3))^(3\cdot 3+1) / (3\cdot 3+1) at x=0 is $-22u_1^3 + 11u_1u_2 - u_3 = A_3^{(3)}$,

in agreement with other methods of calculation, in particular, $[A^{(3)}] = [N][A^{(2)}]$.

III)

Checking

$$A^{(-2)}(x) = \frac{1}{N^{(2)}(x)}$$

$$= \frac{1}{1 + (u_1)x + (2u_1^2 + u_2)x^2 + (5u_1^3 + 6u_2u_1 + u_3)x^3 + (14u_1^4 + 28u_2u_1^2 + 8u_3u_1 + 4u_2^2 + u_4)x^4 + \cdots}:$$

WA gives

series 1 / (1+(u_1) x + (2 u_1^2 + u_2)x^2 + (5 u_1^3 + 6 u_2 u_1 + u_3) x^3 + (14 u_1^4 + 28 u_2 u_1^2 + 8 u_3 u_1 + 4 u_2^2 + u_4)x^4) as

1 - u_1 x + (-u_1^2 - u_2) x^2 + (-2 u_1^3 - 4 u_2 u_1 - u_3) x^3 + (-5 u_1^4 - 15 u_2 u_1^2 - 6 u_3 u_1 - 3 u_2^2 - u_4) x^4 + . . . ,

which gives the first few partition polynomials of $[A^{(-2)}]$ calculated from other methods.

IV)

Checking

$$A_n^{(-2)} = [x^{2n}]\frac{\left(\frac{O(x^2)}{x^2}\right)^{-(-2n+1)}}{-2n+1} = [x^n]\frac{\left(\frac{O(x)}{x}\right)^{-(-2n+1)}}{-2n+1}.$$

WA gives

sixth derivative (1/6!) (1 + u_1 x^2 + u_2 x^4 +u_3 x^6+u_4 x^8)^{-(-6+1)} / (-6+1) at x = 0 is
$-2u_1^3 - 4u_2u_1 - u_3 = A_3^{(-2)}$

and

third derivative (1/3!) (1 + u_1 x + u_2 x^2 +u_3 x^3+u_4 x^4)^{-(-6+1)} / (-6+1) at x = 0 is
$-2u_1^3 - 4u_2u_1 - u_3 = A_3^{(-2)}$.

V)

Checking

$$A_n^{(-2)} = [x^{2n}]\left(\frac{x}{(\frac{x^3}{O(x^2)})^{<-1>}}\right).$$

By the LIF, the nonzero coefficients of $(\frac{x^3}{O(x^2)})^{<-1>}$ are

$$[x^{2n}]\frac{(\frac{O(x^2)}{x^2})^{2n+1}}{2n+1} = [x^n]\frac{(\frac{O(x)}{x})^{2n+1}}{2n+1} = N_n^{(2)}.$$

WA gives

sixth derivative (1/6!) (1 + u_1 x^2 + u_2 x^4 +u_3 x^6+u_4 x^8)^{6+1} / (6+1) at x = 0 is
$5u_1^3 + 6u_2u_1 + u_3 = N_3^{(2)}$

and

third derivative (1/3!) (1 + u_1 x + u_2 x^2 +u_3 x^3+u_4 x^4)^{6+1} / (6+1) at x = 0 is $5u_1^3 + 6u_2 u_1 + u_3 = N_3^{(2)}$,

in agreement with other methods of calculating $[N^{(2)}]$.

Then

$$\frac{x}{(\frac{x^3}{O(x^2)})^{<-1>}} = \frac{x}{x + N_1^{(2)}(u_1)x^3 + N_2^{(2)}(u_1,u_2)x^5 + \cdots}$$

$$= \frac{1}{1 + N_1^{(2)}(u_1)x^2 + N_2^{(2)}(u_1,u_2)x^4 + \cdots},$$

so

$$[x^{2n}]\frac{x}{(\frac{x^3}{O(x^2)})^{<-1>}}$$

$$= [x^{2n}]\frac{1}{1 + N_1^{(2)}(u_1)x^2 + N_2^{(2)}(u_1,u_2)x^4 + \cdots}$$

$$= [x^n]\frac{1}{1 + N_1^{(2)}(u_1)x + N_2^{(2)}(u_1,u_2)x^2 + \cdots} = [x^n]\frac{1}{N^{(2)}(x)},$$

which from check III gives $A_n^{(-2)}$.

VI)

Checking $[N^{(2)}] = [N]^2$.

From several methods of calculation, the first few $[N]$ are

$N_0 = 1$,

$N_1 = u_1$,

$N_2 = u_2 + u_1^2$,

$N_3 = u_3 + 3u_1u_2 + u_1^3$,

$N_4 = u_4 + 4u_1u_3 + 2u_2^2 + 6u_1^2u_2 + u_1^4$,

and the first few $[N^{(2)}]$ are

$N_0^{(2)} = 1$,

$N_1^{(2)} = u_1$,

$N_2^{(2)} = 2u_1^2 + u_2$,

$N_3^{(2)} = 5u_1^3 + 6u_2u_1 + u_3$,

$N_4^{(2)} = 14u_1^4 + 28u_2u_1^2 + 8u_3u_1 + 4u_2^2 + u_4.$

Consistently,

$N_3(u_1, u_2, u_3)|_{u_n \to N_n(u_1,\ldots,u_n)} = (u_3 + 3u_1u_2 + u_1^3)|_{u_n \to N_n(u_1,\ldots,u_n)}$

$= (u_3 + 3u_1u_2 + u_1^3) + 3(u_1)(u_2 + u_1^2) + (u_1)^3$

$= 5u_1^3 + 6u_2u_1 + u_3 = N_3^{(2)}(u_1, u_2, u_3).\backslash$

VII)

Checking $[N^{(-2)}] = [N^{(-1)}]^2 = [N]^{-1}[N]^{(-1}$.

From several different methods of calculation, the first few partition polynomials of $[N]^{-1}$ are

$N_0^{(-1)} = 1$,

$N_1^{(-1)} = u_1$,

$N_2^{(-1)} = u_2 - u_1^2$,

$N_3^{(-1)} = u_3 - 3u_2u_1 + 2u_1^3$,

$N_4^{(-1)} = u_4 - 2u_2^2 - 4u_3u_1 + 10u_2u_1^2 - 5u_1^4$,

and the first few of $[N^{(-2)}]$ are

$N_0^{(-2)} = 1$,

$N_1^{(-2)} = u_1$,

$N_2^{(-2)} = -2u_1^2 + u_2$,

$N_3^{(-2)} = 7u_1^3 - 6u_2u_1 + u_3$,

$N_4^{(-2)} = -30u_1^4 + 36u_2u_1^2 - 8u_3u_1 - 4u_2^2 + u_4.$

Consistently,

$N_3^{(-1)}(u_1, u_2, u_3)|_{u_n \to N_n^{(-1)}(u_1,\ldots,u_n)} = (u_3 - 3u_2u_1 + 2u_1^3)|_{u_n \to N_n^{(-1)}(u_1,\ldots,u_n)}$

$= (u_3 - 3u_2u_1 + 2u_1^3) - 3(u_2 - u_1^2)(u_1) + 2(u_1)^3$

$= 7u_1^3 - 6u_2u_1 + u_3 = N_3^{(-2)}(u_1, u_2, u_3).$

VIII)

Check $[N][A] = [N][A^{(1)}] = [A^{(2)}]$.

The first few $[A]$ are

$A_0 = 1$,

$A_1 = -u_1$,

$A_2 = 2u_1^2 - u_2$,

$A_3 = -5u_1^3 + 5u_1u_2 - u_3$,

$A_4 = 14u_1^4 - 21u_1^2u_2 + 6u_1u_3 + 3u_2^2 - u_4,$

and the first few $[A^{(2)}]$ are

$A_0^{(2)} = 1$,

$A_1^{(2)} = -u_1$,

$A_2^{(2)} = -u_2 + 3u_1^2$,

$A_3^{(2)} = -u_3 + 8u_1u_2 - 12u_1^3$,

$A_4^{(2)} = -u_4 + 10u_1u_3 + 5u_2^2 - 55u_1^2u_2 + 55u_1^4.$

Consistently,

$N_3(u_1, u_2, u_3)|_{u_n \to A_n(u_1,...,u_n)} = (u_3 + 3u_1u_2 + u_1^3)|_{u_n \to A_n(u_1,...,u_n)}$

$= (-5u_1^3 + 5u_1u_2 - u_3) + 3(-u_1)(2u_1^2 - u_2) + (-u_1)^3$

$= -12u_1^3 + 8u_2u_1 - u_3 = A_n^{(2)}(u_1, ..., u_n).$

IX)

Check $[N]^{-1}[A^{(-1)}] = [A^{(-2)}]$.

The first few polynomials of $[N^{(-1)}] = [N]^{-1}$ are

$N_0^{(-1)} = 1$,

$N_1^{(-1)} = u_1$,

$N_2^{(-1)} = u_2 - u_1^2$,

$N_3^{(-1)} = u_3 - 3u_2u_1 + 2u_1^3$,

$N_4^{(-1)} = u_4 - 2u_2^2 - 4u_3u_1 + 10u_2u_1^2 - 5u_1^4$,

the first few polynomials of $[A^{(-1)}]$ ($[K]$ or $[b]$ in other notes) are

$A_0^{(-1)} = 1$,

$A_1^{(-1)} = -u_1$,

$A_2^{(-1)} = -u_2$,

$A_3^{(-1)} = -(u_1u_2 + u_3)$,

$A_4^{(-1)} = -(u_2u_1^2 + 2u_3u_1 + u_2^2 + u_4)$,

and the first few polynomials of $[A^{(-2)}]$ are

$A_0^{(-2)} = 1$,

$A_1^{(-2)} = -u_1$,

$A_2^{(-2)} = -(u_1^2 + u_2)$,

$A_3^{(-2)} = -(2u_1^3 + 4u_2u_1 + u_3)$,

$A_4^{(-2)} = -(5u_1^4 + 15u_2u_1^2 + 6u_3u_1 + 3u_2^2 + u_4)$.

Consistently,

$N_3^{(-1)}(u_1, u_2, u_3)|_{u_n \to A_n^{(-1)}(u_1,\ldots,u_n)} = (u_3 - 3u_2u_1 + 2u_1^3)|_{u_n \to A_n^{(-1)}(u_1,\ldots,u_n)}$

$$= (-(u_1u_2 + u_3)) - 3(-u_2)(-u_1) + 2(-u_1)^3$$

$$= -2u_1^3 - 4u_2u_1 - u_3 = A_3^{(-2)}(u_1, u_2, u_3).$$
