As Above, So Below:
(m)-associahedra and (m)-noncrossing partitions polynomials, Section 1

Tom Copeland, Los Angeles, May 11, 2023

Section 1: Analytic Identities for the Partition Polynomials

For m any integer, the sets $[A^{(m)}]$ of (m)-associahedra partition polynomials (ParPs) $A_n^{(m)}(u_1, u_2, ..., u_n)$ can all be derived recursively from $[A^{(1)}] = [A]$, the re-indexed and factorial re-normalized associahedra ParPs of A133437 with the linear indeterminate set to unity; $[N^{(1)}] = [N]$, the noncrossing partitions ParPs $N_n(u_1, u_2, ..., u_n)$ of A134264; and $[N^{(-1)}] = [N]^{-1}$, the ParPs $N_n^{(-1)}(u_1, u_2, ..., u_n)$ of A350499, the inverse under substitution of indeterminates of $[N]$, via

the substitutional **raising operation** (right into left)

$[A^{(m+1)}] = [N][A^{(m)}]$ for $m > 0$

and the substitutional **lowering operation**

$[A^{(m-1)}] = [N]^{-1}[A^{(m)}]$ for $m < 0$.

For $m = 0$,

$[A^{(0)}] = [R]$,

the reciprocal partition polynomials defined via the formal multiplicative inversion of power series by

$$\frac{1}{1+u_1z+u_2z^2+\cdots} = 1+\sum_{n\geq 1} R_n(u_1,u_2,...,u_n)z^n$$
.

With this inclusion, the ladder operations apply for any integer m.

Dual sets $[N^{(m)}]$ of (m)-noncrossing partitions ParPs $[N^{(m)}]$ can also be defined via iterative substitution using $[N]$ and $[N^{(-1)}]=[N]^{-1}$ simply as

$[N^{(m)}]=[N]^m$.

The first few ParPs of these sets for $m=0,\pm 1,\pm 2,$ and ± 3 are given in a later section, and the multinomial coefficients of any monomial summand of these ParPs are given at the end of this section.

They can be alternatively defined, for $m\neq 0$, as the coefficients of the compositional inverse

$$(O^{(m)}(z))^{(-1)} = z\ (1+A_1^{(m)}(u_1)z^{1m}+A_2^{(m)}(u_1,u_2)z^{2m}+A_3^{(m)}(u_1,u_2,u_3)z^{3m}+ \cdots$$

$$= z\ (1+N_1^{(m)}(R(u_1))z^{1m}+N_2^{(m)}(R_1(u_1),R_2(u_1,u_2))z^{2m}+N_3^{(m)}(R_1(u_1),R_2(u_1,u_2),R_3(u_1,u_2,u_3))z^{3m}+ \cdots$$

of the ‘masked’ power series

$$O^{(m)}(z) = z+u_1\ z^{m+1}+u_2\ z^{2m+1}+u_3\ z^{3m+1}+\cdots .$$

Then the **generalized Lagrange inversion formula** of Lagrange, Schur, and Jabotinsky implies, for m any integer $\neq 0$,

$$A_n^{(m)}(u_1,u_2,...,u_n) = \frac{\partial_{x=0}^{|m|n}}{(n|m|)!}\frac{[1+u_1x^{1|m|}+u_2x^{2|m|}+\cdots+u_nx^{n|m|}]^{-(mn+1)}}{mn+1}$$

$$= \frac{\partial_{z=0}^{n}}{n!}\frac{[1+u_1z+u_2z^2+\cdots+u_nz^n]^{-(mn+1)}}{mn+1}$$

and

$$N_n^{(m)}(u_1,u_2,...,u_n) = \frac{\partial_{x=0}^{n|m|}}{(n|m|)!}\frac{[1+u_1x^{1|m|}+u_2x^{2|m|}+\cdots+u_nx^{n|m|}]^{mn+1}}{mn+1}$$

$$= \frac{\partial_{z=0}^n}{n!} \frac{[1 + u_1 z + u_2 z^2 + \cdots + u_n z^n]^{mn+1}}{mn+1}.$$

The ultimate differential expressions for the partition polynomials imply that

$$A_n^{(0)}(u_1, ..., u_n) = R_n(u_1, ..., u_n),$$

which are the reciprocal partition polynomials described above, and that

$$N_n^{(0)}(u_1, ..., u_n) = u_n,$$

i.e., the partition polynomials for the identity under the substitution operation,

$$[N^{(0)}] = [N]^0 = [I],$$

consistent with the algebraic identities presented below,

The multinomial expansion then allows determination of the monomials of the partition polynomials as presented below.

The definition via compositional inversion implies the **generalized face-h-polynomial substitution identity**, for integer m,

$$[A^{(m)}] = [N^{(m)}][R] = [N^{(m)}][A^{(0)}]$$

and, conversely,

$$[A^{(m)}][R] = [A^{(m)}][A^{(0)}] = [N^{(m)}].$$

I've proven, in older notes not yet posted, an identity for m any integer (stated in different terms for $m > 0$ by Peter Bala in the OEIS, see, e.g., the Formula section of A108767, Oct. 22, 2008) that

$$[N^{(m)}] = [N]^m,$$

so the generalized f-h identity can also be viewed as the **left-sided raising operation**

$$[A^{(m)}] = [N]^m[R] = [N]^m[A^{(0)}] = [N][A^{(m-1)}]$$

or the **left-sided lowering operation**

$[N]^{-1}[A^{(m)}] = [N]^{m-1}[R] = [N]^{m-1}[A^{(0)}] = [A^{(m-1)}]$.

Another perspective from the analysis above, extremely pertinent to combinatorial interpretations, is that $[A^{(m)}]$ and $[N]^m$ are samplings / masks, or are comprised of re-indexed subsets of the monomials, of $[A]$ and $[N]$ for $m > 1$ and of $[A^{(-1)}]$ and $[N]^{-1}$ for $m < 1$; more precisely,

$$P_n^{(m)}(u_1, u_2, ..., u_n) = P_{|m|n}^{sign(m)}(0, \ldots, 0, u_1, 0, \ldots, 0, u_2, 0, \ldots, 0, u_n),$$

where P is either A or N, for $m \neq 0$, with periodically interspersed swathes of $(m-1)$ zeros.

From the characterizations above, $[A^{(m)}]$ is involutive, i.e., with $[I]$ the identity mapping under substitution

$[A^{(m)}] = [A^{(m)}]^{-1}$, or equivalently, $[A^{(m)}]^2 = [I]$,

implying the **right-sided raising operation**

$$[A^{(m)}] = [R][N]^{-m} = [A^{(0)}][N]^{-m} = [A^{(m-1)}][N]^{-1}$$

and the **right-sided lowering operation**

$$[A^{(m)}][N] = [R][N]^{-m+1} = [A^{(0)}][N]^{-m+1} = [A^{(m-1)}].$$

The involution property also implies

1) the **conjugation identity**

$$[A^{(-m)}] = [A^{(0)}][A^{(m)}][A^{(0)}] = [R][A^{(m)}][R]$$

since

$$[R][A^{(m)}][R] = [R][N]^m = ([N]^{-m}[R])^{-1} = ([N]^{-m}[A^{(0)}])^{-1}$$

$$= ([A^{(-m)}])^{-1} = [A^{(-m)}].$$

and

2) the **factorization**

$$[N]^{2m} = [A^{(m)}][A^{(-m)}]$$

since

$$[A^{(m)}][A^{(-m)}] = [N]^m[A^{(0)}][A^{(0)}][N]^m.$$

In particular,

$$[N]^2 = [A^{(1)}][A^{(-1)}] = [A][K],$$

or in terms of the set $[N]^2$ of (2)-noncrossing partitions of OEIS A338135, the set $[A]$ of associahedra polynomials of A133437 (appropriately normalized by the factorial and re-indexed, see examples below), and the special Schur expansion coefficients $[K]$ of A355201

$$[A338135] = [A133437][A355201].$$

$[N]^2$ and $[A]$ have combinatorial interpretations related to certain quantum fields and, therefore, Feynman diagrams.

The **substitutional reciprocal identity**

$$[A^{(m)}] = [R][N]^{-m}$$

also implies

$$A^{(m)}(t) = \frac{1}{N^{(-m)}(t)},$$

with the power series

$$A^{(m)}(t) = 1 + A_1^{(m)}(u_1)t + A_2^{(m)}(u_1, u_2)t^2 + \cdots$$

and

$$N^{(-m)}(t) = 1 + N_1^{(-m)}(u_1)t + N_2^{(-m)}(u_1, u_2)t^2 + \cdots .$$

Coefficients of the ParPs

The ParPs above have $P_0 = 1$ and, for $n = 1, 2, 3, ...$, the generic form

$$P_n(u_1, u_2, ..., u_n) = \sum_{partitions\ of\ n} M(e_1, e_2, ..., e_n) u_1^{e_1} u_2^{e_2} \ldots u_n^{e_3}$$

where $n = 1 \cdot e_1 + 2 \cdot e_2 + \cdots + n \cdot e_n$ and the summation is over all partitions of n as, e.g., presented in the table on p. 831 of "Handbook of Mathematical Functions" by Abramowitz and Stegun. The numerical coefficients $M(e_1, e_2, ..., e_n)$ for the monomial $u_1^{e_1} u_2^{e_2} \ldots u_n^{e_3}$ are, with the sum of exponents $Se = e_1 + e_2 + \cdots + e_n$,

1) $[A]$:

$$\frac{(-1)^{Se}}{n+1} \frac{(n+Se)!}{n! e_1! e_2! \cdots e_n!} = (-1)^{Se} \frac{(n+Se)!}{(n+1)! e_1! e_2! \cdots e_n!} = \frac{1}{n+1} \frac{(-n-1)!}{(-n-1-Se)! e_1! e_2! \ldots e_n!}$$

2) $[A^{(-1)}]$:

$$\frac{-1}{n-1} \frac{(n-1)!}{(n-1-Se)! e_1! e_2! \ldots e_n!} = -\frac{(n-2)!}{(n-Se-1)! e_1! e_2! \ldots e_n!} = \frac{1}{-n+1} \frac{(n-1)!}{(n-1-Se)! e_1! e_2! \ldots e_n!}$$

3) $[N]$:

$$\frac{1}{n+1} \frac{(n+1)!}{(n+1-Se)! e_1! e_2! \ldots e_n!} = \frac{n!}{(n-Se+1)! e_1! e_2! \ldots e_n!}$$

4) $[N]^{-1}$:

$$(-1)^{Se+1} \frac{(n+Se-2)!}{(n-1)! e_1! e_2! \ldots e_n!} = \frac{(-n)!}{(-n-Se+1)! e_1! e_2! \ldots e_n!},$$

where I've incorporated the iconic **rising-falling factorial identity**, central in many discussions of combinatorial reciprocity,

$$\frac{(-x)!}{(-x-j)!} = (-1)^j \frac{(x-1+j)!}{(x-1)!}$$

to identify the reciprocity between the coefficients of $[A]$ and $[A^{(-1)}]$ and between those of $[N]$ and $[N^{(-1)}] = [N]^{-1}$.

Note that $M_{A^{(-1)}} M_{N^{(-1)}} = (-1)^{Se} \frac{1}{(n-1)(n-1+Se)} (\frac{1}{e_1! e_2! \ldots e_n!})^2.$

For the special case case $m = 0$,

$[A^{(0)}] = [R]$, with the first few ParPs

$R_0 = 1$,

$R_1 = -u_1,$

$R_2 = u_1^2 - u_2,$

$R_3 = -u_1^3 + 2u_1 u_2 - u_3,$

$R_4 = u_1^4 - 3u_1^2 u_2 + 2u_1 u_3 + u_2^2 - u_4,$

with the coefficients given by

$(-1)^{Se} \frac{Se!}{e_1! e_2! \ldots e_n!}$.

These are the signed ParPs of A263633, a refined, signed Pascal matrix.

Because of the masking and re-indexing in the compositional inversion relations, for $m \geq 1$ and $n = 1 \cdot e_1 + 2 \cdot e_2 + \cdots + n \cdot e_n$, the numerical coefficients $M(e_1, e_2, ..., e_n)$ for the monomial $u_1^{e_1} u_2^{e_2} \ldots u_n^{e_3}$ are, with $Se = e_1 + e_2 + \cdots + e_n$,

5) $[A^{(m)}]$:

$$\frac{(-1)^{Se}}{mn+1} \frac{(mn+Se)!}{(mn)! e_1! e_2! \cdots e_n!} = (-1)^{Se} \frac{(mn+Se)!}{(mn+1)! e_1! e_2! \cdots e_n!}$$

$$= \frac{1}{mn+1} \frac{(-mn-1)!}{(-mn-1-Se)! e_1! e_2! \ldots e_n!}$$

6) $[A^{(-m)}]$:

$$\frac{-1}{mn-1} \frac{(mn-1)!}{(mn-1-Se)!(e_1)!(e_2)! \ldots (e_n)!} = -\frac{(mn-2)!}{(mn-Se-1)! e_1! e_2! \ldots e_n!}$$

$$= \frac{1}{-mn+1} \frac{(mn-1)!}{(mn-1-Se)!e_1!e_2!\dots e_n!}$$

7) $[N^{(m)}] = [N]^m$:

$$\frac{1}{mn+1} \frac{(mn+1)!}{(mn+1-Se)!e_1!e_2!\dots e_n!} = \frac{(mn)!}{(mn-Se+1)!e_1!e_2!\dots e_n!}$$

8) $[N^{(-m)}] = [N]^{-m}$:

$$(-1)^{Se+1} \frac{(mn+Se-2)!}{(mn-1)!e_1!e_2!\dots e_n!} = \frac{(-mn)!}{(-mn-Se+1)!e_1!e_2!\dots e_n!},$$

This reciprocal identity along with the generalized f-h identity lead to reciprocal relations among the sums of the coefficients and the diagonal coefficients of the dual pair of partition polynomials. Subsets of the absolutes of the diagonal coefficients are sometimes referred to as the (m)-Fuss-Narayana, the (m)-Fuss-Catalan numbers, the (m)-Fuss numbers, or the (m)-positive Catalan numbers. In later notes, I'll indicate by notation that the numbers naturally fall out of the sums or diagonals of the (m)-associahedra and (m)-noncrossing partition ParPs and avoid referring to mathematicians' names, which contain no analytic nor mnemonic value.

It should be apparent by now why I chose the title As Above, So Below--the equations that apply for $m \geq 0$ also apply for $m \leq 0$ with the rising-falling / ascending-descending factorial identity allowing equivalence and also the reciprocal identity applies.
