One Matrix to Rule Them All

Schur self-Konvolution expansion Koefficients; inversion of Laurent and power series; and associahedra, noncrossing, and reciprocal partition polynomials

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“Differential calculus on the Faber polynomials” by Airault and Bouali presents, on p. 186 and 209 (eqns. 1.34 and 6.3), the inverse pair of formal Laurent series

$$g(z) = z + a_1 + \frac{a_2}{z} + \frac{a_3}{z^2} + \cdots$$

and

$$g^{(-1)}(z) = z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \cdots$$

$$= z - a_1 - \sum_{n \geq 1} \frac{K^n_{n+1}}{n} \ \frac{1}{z^n}$$

with, for $n > 0$,

$$-n \ b_{n+1} = K^n_{n+1}(a_1, ..., a_n)$$

$$= \frac{1}{(n+1)!} (\frac{d}{dx})^{n+1} \mid_{x=0} (1 + \sum_{k=0}^{n+1} a_k x^k)^n$$

$$= \frac{1}{(n+1)!} (\frac{d}{dx})^{n+1} \mid_{x=0} (\frac{\hat{f}(x)}{x})^n$$

with $\frac{\hat{f}(x)}{x} = xg(\frac{1}{x}) = 1 + a_1 x + a_2 x^2 + \cdots$.

A & B reference avatars of these relations in "Identities in the theory of power series" by Schur and in the paper “An algebra of differential operators and generating functions on the set of univalent functions” by Airault and Ren. Proofs that $\frac{K^n_{n+1}}{n} = -b_{n+1}$ can be found in both Schur (see eqns. 50, 51, and 57') and A & R---with A & R embedding the more transparent arguments in the text from pages 350-2 with the key ideas centered around equations 1.2.7 and 1.2.8 with $k = n - 1$ in 1.2.8. Schur defined the general self-convolution expansion coefficients K^p_n, with p an integer and $n > 0$ a natural number, for a formal ordinary generating function (o.g.f., or power series) via

$$(Eqn.*)$$

$$(1+\sum_{k\geq 1} a_k x^k)^p = (\frac{\hat{f}(x)}{x})^p = \sum_{k\geq 0} K_k^p(a_1,...,a_k)x^k.$$

Also note

$$(\frac{g(x)}{x})^p = 1+\sum_{n\geq 1} K_k^p \frac{1}{x^k}$$

with $g(x) = x + a_1 + \sum_{n\geq 1} a_{n+1}\frac{1}{z^n}$.

(The Laurent series has the form used in some definitions of Faber polynomials, a phrase I'ii reserve for the related but distinct Faber partition polynomials.)

There are a multitude of identities, differential and otherwise, for these Schur Konvolution Koefficients (a play on K as a mnemonic and the p for its roles as the Power of the defining function and the superscript of K_n^p), which have been presented by Lagrange, Schur, Airault, Bouali, Ren, and others. (See a later section on related general identities presented by Stanley.) Many of the identities are connected to the classical Faber partition polynomials of the Newton-Girard-Waring identities in symmetric function theory (I'll show some of these in two later pdfs). Below I provide some new characterizations of subsets of these Schur coefficients related chiefly to compositional inversion and associated combinatorics.

In addition to characterizing the special involutive subset of Schur self-convolution expansion coefficients $\frac{K_{n+1}^n}{n} = -b_{n+1}$, I'll show how K_{n-1}^{-n} is related to the signed refined face partition polynomials of the associahedra (OEIS A133437, see section B of the answer to the MathOverflow Question "Why is there a connection between enumerative geometry and nonlinear waves?" or my top answer to the MQ-Q "Important formulas in combinatorics") and how K_{n-1}^n is related to the noncrossing partition polynomials of A134264. Then I'll derive identities among these subsets and K_n^{-1}, the reciprocal polynomials giving the coefficients of the o.g.f. of the reciprocal of an ordinary generating function (see the appendix titled "The reciprocal polynomials".. A & B give on p. 185 (eqn. 1.33)

$$\hat{f}^{(-1)}(x) = x + \sum_{n\geq 1} \frac{K_n^{-(n+1)}}{n+1} x^{n+1},$$

but apparently the authors of the papers cited above weren't aware of the links to associahedra nor noncrossing partitions and related combinatorial constructs.

Some sections of this pdf include broader identities for the general Schur expansion coefficients K_n^p, and two forthcoming sets of notes will address more of such via connections with the Faber partition polynomials, leading in particular to a recursion formula for K_n^p given K_m^p for $0 \leq m < n$ that is also noted below and an umbral recursion formula for b_n given the lower order b_m and associated Faber polynomials.

The special Schur self-convolution expansion polynomials b_n

First, let's slightly extend the analysis above by introducing the two additional indeterminates a_0 and b_0 for the linear terms so that

$$g(z) = a_0\ z + a_1 + \frac{a_2}{z} + \frac{a_3}{z^2} + \cdots$$

and

$$g^{(-1)}(z) = b_0\ z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \cdots .$$

{See Appendix (?): *Introducing a general linear coefficient* for details of this extension of the analysis. For the most part, $a_0 = b_0 = 1$ in these notes.)

The following is a list of the first few partition polynomials for b_n in terms of a_n with empirical observations (later proved) on associated combinatorics. Note that the monomials a_1^k are not found in any of the bracketed polynomials after b_1 so that the number of monomial summands is given by A000041 minus one. The remaining partitions are given in the order of those on p. 832 of "Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables" edited by Abramowitz and Stegun.

The first few ***special Schur self-convolution expansion polynomials*** are

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$$b_0 = \frac{1}{a_0}$$

$$b_1 = -\frac{a_1}{a_9}$$

$b_2 = -a_2$

$b_3 = -(a_1 a_2 + a_0 a_3)$

(1,1): coefficients; sum = 2, third Catalan number (A000108)

(1,1): Narayana numbers, second row of A001263, h-vectors of the associahedra; Dyck path / ordered trees numbers, second row of A091869 and A091187.

$b_4 = -(a_1^2 a_2 + a_0 a_2^2 + 2a_0 a_1 a_3 + a_0^2 a_4)$

(1,1,2,1): coefficients: sum = 5, fourth Catalan

(1,1+2=3,1) = (1,3,1): Narayanas--third row from summing the coefficients over the monomial summands with the same order k of a_0^k in increasing order, i.e., the coefficients of the polynomial

$-b_4(a_0 = t, a_1 = 1, a_2 = 1, a_3 = 1, a_4 = 1) = t + 3t^2 + t^3$.

(1,2,1+1) = (1,2,2): summing coefficients over the monomial summands with the same order k of a_1^k in decreasing order , i.e., the coefficients of

$-b_4(a_0 = 1, a_1 = t, a_2 = 1, a_3 = 1, a_4 = 1+ = 2 + 2t + t^2$;

Dyck paths / ordered trees, third row of A091869, reversed A091187.

$b_5 = -(a_1^3 a_2 + 3a_0 a_1 a_2^2 + 3a_0 a_1^2 a_3 + 3a_0^2 a_2 a_3 + 3a_0^2 a_1 a_4 + a_0^3 a_5)$.

(1,3,3,3,3,1), coefficients; sum = 14, fifth Catalan

(1,6,6,1), Narayanas--fourth row

(1, 3, 3+3, 3+1) = (1,3,6,4):

$$-b_5(1_0 = 1, a_1 = t, a_2 = 1, ..., a_5 = 1) = 4 + 6t + 3t^2 + t^4;$$

Dyck paths / ordered trees, fourth row of A091869, reversed A091187.

$$b_6 = -(a_1^4 a_2 + 6a_0 a_1^2 a_2^2 + 4a_0 a_1^3 a_3 + 2a_0^2 a_2^3 + 12a_0^2 a_1 a_2 a_3$$

$$+6a_0^2 a_1^2 a_4 + 2a_0^3 a_3^2 + 4a_0^3 a_2 a_4 + 4a_0^3 a_1 a_5 + a_0^4 a_6)$$

(1,6,4,2,12,6,2,4,4,1): coefficients; sum = 42, sixth Catalan

(1,10,20,10,1) Narayanas, fifth row

(1,4,6+6,12+4,2+2+4+1) = (1,4,12,16,9);

Dyck paths / ordered trees, fifth row of A091187 and reversed A091869.

$$b_7 = -(a_1^5 a_2 + 10a_0 a_1^3 a_2^2 + 5a_0 a_1^4 a_3 + 10a_0^2 a_1 a_2^3 + 30a_0^2 a_1^2 a_2 a_3$$

$$+10a_0^2 a_1^3 a_4 + 10a_0^3 a_2^2 a_3 + 10a_0^3 a_1 a_3^2 + 20a_0^3 a_1 a_2 a_4$$

$$+10a_0^3 a_1^2 a_5 + 5a_0^4 a_3 a_4 + 5a_0^4 a_2 a_5 + 5a_0^4 a_1 a_6 + a_0^5 a_7)$$

(1,10,5,10,30,10,10,10,20,10,5,5,5,1): coefficients, sum = 132, seventh Catalan

(1, 15, 50, 50, 15, 1): Narayanas--sixth row

(1,5,20,40,45,21): Dyck paths / ordered trees, fifth row of A091187 and reversed A091869.

Let’s now prove what the numerical evidence above suggests; the partition polynomials for Schur’s self-convolution coefficients are intimately related to the partition polynomials reflecting the combinatorics of the associahedra and the noncrossing partitions (and their numerous associated combinatorial constructs, such as trees, lattice paths, and quivers).

Compositional and multiplicative inversions, Laurent series, and the noncrossing partitions and special Schur expansion coefficients b_n

Following the analysis in my notes and MO-Q on the Schwarz-Kac ops and Alexandrov's paper (p. 21) and identifying c_n below with a_n above, the inverse of the Laurent series

$$g(x) = LC(1/x) = x + c_1 + \frac{c_2}{x} + \frac{c_3}{x^2} + \cdots$$

is

$$g^{(-1)}(x) = \frac{1}{LC^{(-1)}(x)}$$

$$= \frac{1}{\frac{1}{x} + \frac{NCP_1(c_1)}{x^2} + \frac{NCP_2(c_1,c_2)}{x^3} + \frac{NCP_3(c_1,\ldots,c_3)}{x^4} + \frac{NCP_4(c_1,\ldots,c_4)}{x^5} + \ldots}$$

$$= \frac{1}{\frac{1}{x} + \frac{c_1}{x^2} + \frac{c_2+c_1^2}{x^3} + \frac{c_3+3c_2c_1+c_1^3}{x^4} + \frac{c_4+4c_1c_3+2c_2^2+6c_1^2c_2+c_1^4}{x^5} + \ldots}$$

$$= x - c_1 - \frac{c_2}{x} - \frac{c_1c_2+c_3}{x^2} - \frac{c_1^2c_2+2c_1c_3+c_2^2+c_4}{x^3} - \cdots$$

$$= x + b_1 + \frac{b_2}{x} + \frac{b_3}{x^2} + \frac{b_4}{x^3} + \cdots ,$$

where the NCP_n are the partition polynomials of OEIS A134264, the inversion polynomials giving the coefficients of the formal o.g.f. $O^{(-1)}(x)$ that is the compositional inverse of a formal o.g.f. $O(x)$ in terms of the coefficients h_n of its shifted reciprocal $h(x) = x/O(x) = \sum_{n\geq 0} h_n x^n$. This inverse agrees with that of A & B with $a_n = c_n$ and $a_0 = h_0 = 1$. Note from the comments, links, and refs in the OEIS that the NCP_n have numerous combinatorial interpretations, including labeling and enumerating noncrossing partitions (NCP).

Now form

$$= x\ g^{(-1)}(1/x) = \frac{1}{1+NCP_1x+NCP_2x^2+NCP_3x^3+NCP_4x^4+\cdots}$$

$$= \frac{1}{1+c_1x+(c_2+c_1^2)x^2+(c_3+3c_2c_1+c_1^3)x^3+(c+4+4c_1c_3+2c_2^2+6c_1^2c_2+c_1^4)x^4+\cdots}$$

$= 1 - c_1x - c_2x^2 - (c_1c2 + c_3)x^3 - (c_1^2c_2 + 2c_1c_3 + c_2^2 + c_4)x^4 - \cdots$

$= 1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 + \cdots$

$= \frac{x}{f^{(-1)}(x)}$,

where, from the inversion interpretation of the partition polynomials of A134264,

$f^{(-1)}(x) = x + NCP_1x^2 + NCP_2x^3 + NCP_3x^4 + NCP_4x^5 + \cdots$

$= x + c_1x^2 + (c_2 + c_1^2)x^3 + (c_3 + 3c_2c_1 + c_1^3)x^4$

$+ (c_4 + 4c_1c_3 + 2c_2^2 + 6c_1^2c_2 + c_1^4)x^5 + \cdots$

and

$h(x) = \frac{x}{f(x)} = 1 + c_1x + c_2x^2 + c_3x^3 + \cdots = x\ g(1/x).$

Then

$f(x) = \frac{x}{h(x)} = \frac{1}{g(1/x)} = \frac{x}{1+c_1x+c_2x^2+c_3x^3+\ldots}$

and

$g(x) = x\ h(1/x) = \frac{1}{f(1/x)} = x + c_1 + \frac{c_2}{x} + \frac{c_3}{x^2} + \cdots$.

The Schur expansion coefficients, the associahedra and noncrossing partition polynomials and compositional and multiplicative inversions

Let's relate the analysis to the Schur self-Konvolution expansion coefficients K_n^p defined as in $(Eqn. *)$ by

$(1 + \sum_{k\geq 1} c_kx^k)^p = \sum_{k\geq 0} K_k^p(c_1, ..., c_n)x^k.$

Define the reciprocals

$1+\sum_{k\geq 1} c_k x^k = h(x) = \frac{x}{f(x)} = \frac{\hat{f}(x)}{x} = \frac{1}{\hat{h}(x)} = \frac{1}{1+\sum_{k\geq 1}\hat{c}_k x^k}.$

Then

$(\frac{x}{f(x)})^p = (h(x))^p = (1+\sum_{k\geq 1} c_k x^k)^p = \sum_{k\geq 0} K_k^p(c_1,...,c_k)x^k$

$= (\frac{\hat{f}(x)}{x})^p = (\frac{1}{\hat{h}(x)})^p = (\hat{h}(x))^{-p} = (1+\sum_{k\geq 1}\hat{c}_k x^k)^{-p}$

$= \sum_{k\geq 0} K_k^{-p}(\hat{c}_1,...,\hat{c}_k)x^k,$

identifying

$K_n^p(c_1,...,c_n) = K_n^{-p}(\hat{c}_1,...,\hat{c}_n).$

The signed refined face partition polynomials, or refined Euler characteristic polynomials, of the associahedra, $ASP_n(a_1,..a_n)$, presented in A133437 give the coefficients of the o.g.f. $f^{(-1)}(x)$, the formal compositional inverse about the origin of the formal o.g.f. $f(x) = x + a_1 x + a_2 x^2 + ...$, and the noncrossing partition polynomials of A134264, $NCP_n(c_1,...,c_n)$, give the same comp. Inverse, as indicated above, but in terms of the coefficients of the o.g.f. that is the shifted reciprocal of $f(x)$, i.e., $h(x) = \frac{x}{f(x)} = 1 + c_1 x + c_2 x^2 + ...$.

Then the classic Lagrange inversion theorem/formula (LIF) implies

$\frac{D_{x=0}^{n-1}}{(n-1)!}\frac{(h(x))^n}{n} = \frac{D_{x=0}^{n-1}}{(n-1)!}\frac{(\hat{h}(x))^{-n}}{n} = \frac{D_{x=0}^{n-1}}{(n-1)!}\frac{(\frac{x}{f(x)})^n}{n} = \frac{D_{x=0}^{n-1}}{(n-1)!}\frac{(\frac{\hat{f}(x)}{x})^n}{n} = \frac{D_{x=0}^{n}}{n!} f^{(-1)}(x)$

$= ASP_n(\hat{c}_1,...,\hat{c}_n) = NCP_n(c_1,...c_n) = \frac{K_{n-1}^n(c_1,...c_{n-1})}{n} = \frac{K_{n-1}^{-n}(\hat{c}_1,...,\hat{c}_{n-1})}{n}$

and, conversely,

$\frac{D_{x=0}^{n-1}}{(n-1)!}\frac{(\hat{h}(x))^n}{n} = \frac{D_{x=0}^{n-1}}{(n-1)!}\frac{(h(x))^{-n}}{n} = \frac{D_{x=0}^{n-1}}{(n-1)!}\frac{(\frac{f(x)}{x})^n}{n} = \frac{D_{x=0}^{n-1}}{(n-1)!}\frac{\left(\frac{x}{\hat{f}(x)}\right)^n}{n} = \frac{D_{x=0}^{n}}{n!} \hat{f}^{(-1)}(x)$

$= ASP_n(c_1,...c_n) = NCP_n(\hat{c}_1,...,\hat{c}_n) = \frac{K_{n-1}^n(\hat{c}_1,...,\hat{c}_{n-1})}{n} = \frac{K_{n-1}^{-n}(c_1,...,c_{n-1})}{n}.$

FInally, we have from Schur, as noted above, that

$$\frac{D_{x=0}^{n+1}}{(n+1)!}\frac{(h(x))^n}{n} = \frac{D_{x=0}^{n+1}}{(n+1)!}\frac{(\frac{x}{f(x)})^n}{n}$$

$$= \frac{D_{x=0}^{n+1}}{(n+1)!}\frac{(\frac{\hat{f}(x)}{x})^n}{n} = \frac{K_{n+1}^n}{n}(c_1,...,c_{n+1}) = -b_{n+1}(c_1,...c_{n+1}).$$

Note for computational purposes that

$$K_n^p(c_1,...,c_n) = \frac{D_{x=0}^n}{n!}\sum_{k\geq 1}\ K_m^p x^m$$

$$= \frac{D_{x=0}^n}{n!}\ (1+\sum_{k\geq 1} c_k x^k)^p = \frac{D_{x=0}^n}{n!}\ (1+\sum_{k=1}^n c_k x^k)^p,$$

where the last sum has a finite number of terms, ensuring the analysis is graded and, therefore, applies to formal o.g.f.s as well as convergent ones.

Identities among the reciprocal, associahedra, noncrossing, and the special Schur self-convolution expansion partition polynomials

Reprising, the o.g.f.s in the formulas for computing the Schur coefficients are

$$h(x) = \frac{x}{f(x)} = \frac{\hat{f}(x)}{x} = xg(\tfrac{1}{x}) = 1 + c_1x + c_2x^2 + c_3x^3 + \cdots,$$

$$f(x) = \frac{x}{h(x)} = \frac{x^2}{\hat{f}(x)} = \frac{x}{1+c_1x+c_2x^2+c_3x^3+\cdots}$$

$$= x + \hat{c}_1x^2 + \hat{c}_2x^3 + \hat{c}_3x^4 + \cdots,$$

$$\hat{f}(x) = x\ h(x) = x + c_1x^2 + c_2x^3 + c_3x^4 + \cdots.$$

The coefficients of the o.g.f. $f(x)$ are determined in terms of the c_n by the signed reciprocal partition polynomials R_n of A263633 (see the appendix below titled ‘The reciprocal polynomials”.. Then its inverse o.g.f. $f^{(-1)}(x)$ can be determined either with associahedra polynomials $ASP_n(\hat{c}_1,\hat{c}_2,...,\hat{c}_n)$ of A133437 with the indeterminates $R_n(c_1,...,c_n) = \hat{c}_n$ or with the noncrossing partition polynomials $NCP_n(c_1, c_2, ..., c_n)$ of A134264 with the indeterminates c_n. This is a verbalization of the identity

$$f^{(-1)}(x) = x + \sum_{n \geq 2} ASP_n[R_1(c_1), ..., R_n(c_1, ..., c_n)]$$

$$= x + \sum_{n \geq 2} NCP_n[c_1), ..., c_n]$$,

and we can identify

$$NCP_n[c_1), ..., c_n] = ASP_n[R_1(c_1), ..., R_n(c_1, ..., c_n)],$$

which I'll express more concisely as the substitution/composition identity

$$[NCP] = [ASP][R]$$

for the three sets of partition polynomials.

In addition, the coefficients of the inverse o.g.f. $\hat{f}^{(-1)}(x)$ are determined by the $ASP_n(c_1, c_2, ..., c_)$ or by $NCP_n(\hat{c}_1, \hat{c}_2, ..., \hat{c}_n)$; that is,

$$\hat{f}^{(-1)}(x) = x + \sum_{n \geq 2} ASP_n[c_1, ...c_n]$$

$$= x + \sum_{n \geq 2} NCP_n[R_1(c_1), ..., R_n(c_1, ..., c_n)],$$

and we can identify

$$ASP_n[c_1, ...c_n] = NCP_n[R_1(c_1), ..., R_n(c_1, ..., c_n)],$$

and conclude

$$[ASP] = [NCP][R]$$,

consistent with $[R]^2 = [I]$, the identity transformation under substitution.

Consequently,

$$x \; g^{(-1)}(1/x) = \frac{x}{f^{(-1)}(x)}$$

$$= \frac{1}{1+NCP_1(c_1)x+NCP_2(c_1,c_2)x^2+NCP_3(c_1,c_2,c_3)x^3+NCP_4(c_1,c_2,c_3,c_4)x^4+\cdots}$$

$$= \frac{1}{1+c_1x+(c_2+c_1^2)x^2+(c_3+3c_2c_1+c_1^3)x^3+(c_4+4c_1c_3+2c_2^2+6c_1^2c_2+c_1^4)x^4+\cdots}$$

$$= 1 - c_1x - c_2x^2 - (c_1c2 + c_3)x^3 - (c_1^2c_2 + 2c_1c_3 + c_2^2 + c_4)x^4 - \cdots$$

$$= 1 + b_1x + b_2x^2 + b_3x^3 + b_4x^4 - \cdots$$

$$= 1 - c_1x - \frac{K_2^1(c_1,c_2)}{2}x^2 - \frac{K_3^2(c_1,c_2,c_3)}{3}x^3 - \frac{K_4^3(c_1,c_2,c_3,c_4)}{4}x^4 - \cdots$$

$$= 1 + R_1(NCP_1(c_1))x + R_2(NCP_1(c_1), NCP_2(c_1,c_2))x^2$$

$$+R_3(NCP_1, ..., NCP_3)x^3 + R_4(NCP_1, ..., NCP_4)x^4 - \cdots ,$$

where the reciprocal polynomials R_n for o.g.f.s (A263633 mod signs) are determined by

$$1 + \sum_{n\geq 1} R_n(d_1, ..., d_n)x^n = \frac{1}{1+d_1x+d_2x^2+d_3x^3+...}$$

$$= 1 + (-d_1)x + (d_1^2 - d_2)x^2 + (-d_1^3 + 2d_1d_2 - d_3)x^3 + (d_1^4 - 3d_1^2d_2 + 2d_1d_3 + d_2^2 - d_4)x^4$$

$$+(-d_1^5 + 4d_1^3d_2 - 3d_1^2d_3 + 2d_1d_4 - 3d_1d_2^2 + 2d_2d_3 - d_5)x^5 + \ldots .$$

We can distill from these identities that

$$[b] = [R][NCP].$$

Because the reciprocal is an involution and

$R_n[NCP_1(c_1), ..., NCP_n(c_1, ..., c_n)] = b_n(c_1, ..., c_n)$,

then

$$R_n[b_1(c_1), ..., b_n(c_1, ..., c_n)] = NCP_n(c_1, ..., c_n),$$

or

$[R][b] = [NCP].$

Repeating, once again, arguments above, since

$$f(x) = \frac{x}{h(x)} = \frac{x^2}{\hat{f}(x)} = \frac{x}{1+c_1x+c_2x^2+c_3x^3+\cdots} \quad = x + R_1(c_1)x^2 + R_2(c_1,c_2)x^3 + ...,$$

then

$$f^{(-1)}(x) = x + NCP_1(c_1)x^2 + NCP_2(c_1,c_2)x^3 + ...$$

$$= x + ASP_1[R_1(c_1)]x^2 + ASP_2[R_1(c_1), ..., R_2(c_1, ..., c_n)]x^3 + \dots$$

and

$$NCP_n(c_1, ..., c_n) = ASP_n[R_1(c_1), ..., R_n(c_1, ..., c_n)].$$

Since also from above,

$$R_n[b_1(c_1), ..., b_n(c_1, ..., c_n)] = NCP_n(c_1, ..., c_n)$$

and the reciprocal polynomials form an involutive set,

$$b_n(c_1, ..., c_n) = R_n[NCP_1(c_1), \dots, NCP_n(c_1, ..., c_n)]$$

$$= R_n[ASP_1[R_1(c_1)], \dots, ASP_n[R_1(c_1), ..., R_n(c_1, ..., c_n)]].$$

More concisely, we have the conjugation

$$[b] = [R][NCP] = [R][ASP][R] = .[R]^{-1}[ASP][R] = [R][ASP][R]^{-1}.$$

Reprising and iterating the basic Identities among the reciprocal, associahedra, noncrossing, and the special Schur self-convolution expansion partition polynomials

Reprising for brevity,

$$[R] = [R]^{-1},$$

$[ASP] = [ASP]^{-1}$,

$[b] = [R][NCP]$,

$[R][b] = [NCP]$,

$[NCP] =]ASP][R]$,

$[ASP] = [NCP][R]$.

Combining the identities,

$[ASP][b] = [NCP][R][b] = [NCP][NCP]$ and, consistently,

$[b] = [ASP]NCP][R][b] = [R][R][b] = [b] = [ASP][NCP][NCP] = [R][NCP]$.

We can identify the conjugation, or 'similarity', transformation

$[b] = [R][ASP][R] = [R]^{-1}[ASP][R] = [R][ASP][R]^{-1}$,

an isomorphism between an involutive Lagrange inversion set of partition polynomials $[ASP]$ for compositional inverse pairs of o.g.f.s and an involutive Lagrange inversion set of partition polynomials $[b]$ for compositional inverse pairs of particular Laurent series.

Note that the five pairs of partition polynomials

$[R]$ and $[ASP]$, $[R]$ and $[NCP]$, $[R]$ and $[b]$, $[ASP]$ and $[NCP]$, or $[b]$ and $[NCP]$ are sufficient to generate the remaining two sets, The pair $[ASP]$ and $[b]$ is not sufficient to do--they can generate even-order self-compositions of $[NCP]$. $[NCP]$ is the only non-involutive set. (For completeness, examples of partition polynomials generated by $[b][ASP]$ are in one of the last sections of this pdf.)

We also have the miscellaneous relations, derived from the above,

$[NCP]^{2n} = ([ASP][b])^n$,

$[R][b][R] = [NCP][R] = [ASP]$,

$[b][R] = [R][ASP] = [R][NCP][R]$,

$([b][R])^n = R[NCP]^n R$,

$[NCP]^n = R([b][R])^n R = ([R][b])^n = ([ASP][R])^n$.

My MathOverflow question "Combinatorics of iterated composition of noncrossing partition polynomials" and blog post "Matryoshka Dolls: Iterated noncrossing partitions, the refined Narayana group, and quantum fields" presents more info on the $[NCP]^m = [N]^m$ which I call the refined $m-$ Narayana polynomials for m any integer, a group with the two generators $[NCP] = [N]$ and its inverse set of partition polynomials $[N]^{-1}$ under iterated substitution, or composition. The blog post also presents the parallel e.g.f. / Taylor series version--the refined Eulerian group. (More on these groups in forthcoming pdfs.)

The direct bijections with Shur self-convolution expansion coefficients are

$R \leftrightarrow K_n^{-1}$,

$ASP \leftrightarrow \frac{K_{n-1}^n}{n}$,

$b \leftrightarrow \frac{K_n^{n-1}}{n-1}$.

The general Schur self-convolution expansion coefficients K_n^p encompass the important partition polynomials characterizing the various avatars of Lagrange inversion of o.g.f.s.--one array to rule them all.

A change of variables in the equations giving the inverse of $g(x)$ leads to a slight variation in the arguments above and the same basic identities among the partition polynomials:

$$w(x) = x^2\, g^{(-1)}(1/x) = x + b_1(c_1)x^2 + b_2(c_1,c_2)x^3 + b_3(c_1,c_2,c_3)x^4 + \cdots$$

$$= x^2 \frac{1}{LC^{(-1)}(1/x)} = \frac{x}{1+NCP_1(c_1)x+NCP_2(c_1,c_2)x^2+NCP_3(c_1,\ldots,c_3)x^3+\ldots},$$

so we can use either the ASP_n to determine the compositional inverse of $w(x)$ or the NCP_n leading once again to the identities

$ASP_n[b_1(c_1), ..., b_n(c_1, ..., c_n)] = NCP_n[NCP_1(c_1), ..., NCP_n(c_1, ..., c_n)]$

$= ASP_n[R_1(NCP_1(c1)), ..., R_n(NCP_1(c_1), ..., NCP_n(c_1, ..., c_n))]$

and

$b_n(c_1, ..., c_n) = ASP_n[NCP_1[NCP_1(c_1)], ..., NCP_n[NCP_1(c_1), ..., NCP_n(c_1, ...c_1)]],$

confirming

$[ASP][b] = [NCP][NCP]$

and leading, via $[ASP]^2 = I$, to the variation

$[b] = [R][NCP]$.

Spot-checks:

For easy reference, with $c_0 = 1$ and an obvious abbreviation of notation, the first few partition polynomials of the special set of Schur expansion polynomials are

$b_1 = -c_1$,

$b_2 = -c_2$,

$b_3 = -(c_1c_2 + c_3)$,

$b_4 = -(c_1^2c_2 + 2c_1c_3 + c_2^2 + c_4)$,

$b_5 = -(c_1^3c_2 + 3c_1c_2^2 + 3c_1^2c_3 + 3c_2c_3 + 3c_1c_4 + c_5)$,

the first few refined associahedra Euler characteristic polynomials are

$A_1(u_1) = -u_1$,

$A_2(u_1, u_2) = 2u_1^2 - u_2$,

$A_3(u_1, u_2, u_3) = -5u_1^3 + 5u_1u_2 - u_3$,

$A_4(u_1, u_2, u_3, u_4) = 14u_1^4 - 21u_1^2u_2 + 6u_1u_3 + 3u_2^2 - u_4$,

and

$N_1(u_1) = u_1$,

$N_2(u_1, u_2) = u_1^2 + u_2$,

$N_3(u_1, u_2, u_3) = u_3 + 3u_2u_1 + u_1^3$.

First check:

$A_1[b_1] = -(-c_1) = c_1$,

and

$N_1[N_1(c_1)] = c_1$

Second check:

$A_2[b_1, b_2] = 2(-c_1)^2 - (-c_2) = 2c_1^2 + c_2$

and

$N_2[N_1(c_1), N_2(c_1, c_2)] = u_2 + u_1^2 \mid_{u_n = N_n(c_1, \ldots, c_n)}$

$= c_2 + c_1^2 + c_1^2 = c_2 + 2c_1^2$, so

$A_2[N_1[N_1(c_1)], N_2[N_1(c_1), N_2(c_1, c_2)]]$

$= A_2[c_1, c_2 + 2c_1^2] = 2c_1^2 - (c_2 + 2c_1^2) = -c_2 = b_2(c_1, c_2)$.

Third order check:

$A_3[b_1(c_1), b_2(c_1, c_2), b_3(c_1, c_2, c_3)] = -5b_1^3 + 5b_1b_2 - b_3$

$= 5c_1^3 + 5c_1c_2 + c_1c_2 + c_3 = 5c_1^3 + 6c_1c_2 + c_3$

and

$N_3[N_1(c_1), N_2(c_1, c_2), N_3(c_1, c_2, c_3)]$

$= (c_3 + 3c_2c_1 + c_1^3) + 3(c_1^2 + c_2)(c_1) + c_1^3 = c_3 + 6c_1c_2 + 5c_1^3$,

which is the third row of A338135.

Fourth order check:

$A_4[b_1(c_1), b_2(c_1, c_2), b_3(c_1, c_2, c_3), b_4(c_1, ..., c_4)]$

$= 14b_1^4 - 21b_1^2b_2 + 6b_1b_3 + 3b_2^2 - b_4$

$= 14c_1^4 + 21c_1^2c_2 + 6(c_1)(c_1c_2 + c_3) + 3c_2^2 + (c_1^2c_2 + 2c_1c_3 + c_2^2 + c_4)$

$= 14c_1^4 + 28c_2c_1^2 + 8c_3c_1 + 4c_2^2 + c_4$,

which is the fourth row of A338135.

Fifth order check:

$A_5(b_1, ..., b_5) = -42b_1^5 + 84b_1^3b_2 - 28b_1b_2^2 - 28b_1^2b_3 + 7b_1b_4 + 7b_2b_3 - b_5$

$= 42c_1^5 + 84c_1^3c_2 + 28c_1c_2^2 + 28c_1^2(c_1c_2 + c_3) + 7c_1(c_1^2c_2 + 2c_1c_3$

$+ c_2^2 + c_4) + 7c_2(c_1c_2 + c_3) + (c_1^3c_2 + 3c_1c_2^2 + 3c_1^2c_3 + 3c_2c_3 + 3c_1c_4 + c_5)$

$= 42c_1^5 + 120c_2c_1^3 + 45c_3c_1^2 + 45c_2^2c_1 + 10c_4c_1 + 10c_2c_3 + c_5$,

which is the fifth row of A338135.

More illustrations and checks:

$\frac{(h(x))^1}{1} = (1 + c_1x + c_2x^2 + c_3x^3 + c_4x^4)^1/1 = 1 + c_1x + c_2x^2 + ...,$

in which the coefficient of x^2 is $\frac{K_2^1}{1} = -b_2 = c_2$.

$\frac{(h(x))^3}{3} = (1 + c1x + c_2x^2 + c_3x^3 + c_4x^4)^3 \ / \ 3$

$= 1/3 + c_1x + (c_1^2 + c_2)x^2 + (c_1^3/3 + 2c_2c_1 + c_3)x^3 + (c_2c_1^2 + 2c_3c_1 + c_2^2 + c_4)x^4 + ...,$

in which the coefficient of x^4 is $\frac{K_4^3}{3} = -b_4$.

$R_2(NCP_1(c_1), NCP_2(c_1, c_2)) = (d_1^2 - d_2)|_{d_n = NCP_n}$

$= (NCP_1(c_1))^2 - NCP_2(c_1, c_2) = (c_1)^2 - (c_2 + c_1^2) = -c_2 = b_2.$

$R_3(NCP_1(c_1), NCP_2(c_1, c_2), NCP_3(c_1, ...)) = (-d_1^3 + 2d_1d_2 - d_3)\ |_{d_n = NCP_n}$

$= -c_1^3 + 2c_1(c_2 + c_1^2) - (c_3 + 3c_2c_1 + c_1^3) = -(c_2c_1 + c_1^3) = b_3.$

$R_2[b_1, ..., b_2] = (d_1^2 - d_2)\ |_{d_n = b_n} = (-c_1)^2 - (-c_2) = c_1^2 + c_2 = NCP_2(c_1, c_2).$

$R_3[b_1, ..., b_2] = (-d_1^3 + 2d_1d_2 - d_3)\ |_{d_n = b_n}$

$= -(-c_1)^3 + 2(-c_1)(-c_2) - (-(c_1c2 + c_3)) = c_1^3 + 3c_1c_2 + c_3 = NCP_3(c_1, c_2, c_3).$

The first few associahedra polynomials ASP_n of A133437 with a shift in indices and $u_0 = 1$ are

$ASP_0 = 1$,

$ASP_1 = -u_1$,

$ASP_2 = 2u_1^2 - u_2$,

$ASP_3 = -5u_1^3 + 5u_1u_2 - u_3,$

so

$ASP_2[R_1(c_1), ..., R_2(c_1, c_2)] = (2u_1^2 - u_2)\ |_{u_n = R_n}$

$$= 2(-c_1)^2 - (c_1^2 - c_2) = c_1^2 + c_2 = NCP_2(c_1, c_2)$$

and

$$ASP_3[R_1(c_1), ..., R_3(c_1, ..., c_3)] = (-5u_1^3 + 5u_1u_2 - u_3)\mid_{u_n = R_n}$$

$$= -5(-c_1)^3 + 5(-c_1)(c_1^2 - c_2) - (-c_1^3 + 2c_1c_2 - c_3)$$

$$= c_1^3 + 3c_1c_2 + c_3 = NCP_3(c_1, ..., c_3).$$

The set of Schur coefficients $\frac{K_{n-1}^n}{n}$ have an interpretation as the coefficients of the o.g.f. of the inverse $f^{(-1)}(x)$ of $f^(x)$ about the origin in terms of the coefficients of the shifted reciprocal

$$\frac{x}{f(x)} = h(x) = 1 + c_1x + c_2x^2 + c_3x^3 + \cdots = x\ g(1/x)$$

since, by the Lagrange inversion formula,

$$\frac{K_{n-1}^n}{n} = \frac{D_{x=0}^{n-1}}{(n-1)!} \frac{(\frac{x}{f(x)})^n}{n} = NCP_n(c_1, ..., c_n).$$

Checks:

$$(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6)^4/4$$

$$= \frac{c_0^4}{4} + c_0^3c_1x + (c_2c_0^3 + \tfrac{3}{2}c_1^2c_0^2)x^2 + (c_0c_1^3 + 3c_0^2c_2c_1 + c_0^3c_3)x^3$$

$$+ (\tfrac{c_1^4}{4} + 3c_0c_2c_1^2 + 3c_0^2c_3c_1 + \tfrac{3}{2}c_0^2c_2^2 + c_0^3c_4)x^4$$

$$+ (c_2c_1^3 + 3c_0c_3c_1^2 + 3c_0c_1c_2^2 + 3c_0^2c_1c_4 + 3c_0^2c_2c_3 + c_0^3c_5)x^5 + \cdots .$$

The coefficient of the third order term is

$NCP_3(c_0,...,c_3)) = (c_0c_1^3 + 3c_0^2c_2c_1 + c_0^3c_3)$,

the fourth order coefficient of the o.g.f. for $f^{(-1)}(x)$ per A134264.

$$(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6)^5/5$$

$$= c_0^5/5 + c_0^4c_1x + (2c_0^3c_1^2 + c_0^4c_2)x^2 + (2c_0^2c_1^3 + 4c_0^3c_2c_1 + c_0^4c_3)x^3$$

$$+ (c_0^4c_4 + 2c_0^3c_2^2 + 4c_0^3c_1c_3 + 6c_0^2c_1^2c_2 + c_0c_1^4)x^4$$

$$+ (\tfrac{c_1^5}{5} + 4c_0c_2c_1^3 + 6c_0^2c_3c_1^2 + 6c_0^2c_1c_2^2 + 4c_0^3c_1c_4 + 4c_0^3c_2c_3 + c_0^4c_5)x^5$$

$$+ (c_1^4c_2 + 6c_0c_1^2c_2^2 + 4c_0c_1^3c_3 + 2c_0^2c_2^3 + 12c_0^2c_1c_2c_3$$

$$+ 6c_0^2c_1^2c_4 + 2c_0^3c_3^2 + 4c_0^3c_2c_4 + 4c_0^3c_1c_5 + c_0^4c_6)x^6 + \cdots$$

The coefficient of the fourth order term is

$$NCP_4(c_0,...,c_4) = c_0^4c_4 + 2c_0^3c_2^2 + 4c_0^3c_1c_3 + 6c_0^2c_1^2c_2 + c_0c_1^4)$$

per A134264, the coefficient of the fifth order term of $f^{(-1)}(x)$.

Reductions, or specializations, of the special Schur self-convolution expansion coefficients b_n

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Reduction to the Catalan numbers (A000108):

The following analysis reveals that the sums of the coefficients of the partition polynomials b_n are the celebrated Catalan numbers.

An o.g.f. for the Catalans is

$$Cat(x) = \frac{1-\sqrt{1-4x}}{2} = \sum_{n\geq 1} Cat_n x^n$$

$$= x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + \cdots$$

with inverse

$$Cat^{-1}(x) = x(1-x) = x - x^2$$

Let

$$f(x) = Cat^{-1}(x) = x(1-x).$$

Then

$$f^{(-1)}(x) = Cat(x) = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + ...$$

and

$$h(x) = \frac{x}{f(x)} = \frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n\geq 0} c_n x^n.$$

Consequently, the general formulas give

$$g(x) = x\ h(1/x) = \frac{1}{f(1/x)} = x + c_1 + \frac{c_2}{x} + \frac{c_3}{x^2} + \cdots$$

$$= x\frac{1}{1-\frac{1}{x}} = x + 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots.$$

$$xg^{(-1)}(1/x) = \frac{x}{f^{-1}(x)} = \frac{x}{Cat(x)} = \frac{x}{1-\sqrt{1-4x}} = \frac{1+\sqrt{1-4x}}{2}$$

$$= 1 - x - x^2 - 2x^3 - 5x^4 - 14x^5 + ... = 1 - \sum_{n\geq 1} Cat_n x^n.$$

$$g^{(-1)}(x) = \frac{1}{f^{(-1)}(1/x)} = \frac{2}{1-\sqrt{1-4/x}} = x\frac{1+\sqrt{1-4/x}}{2}$$

$$= x - 1 - \frac{1}{x} - \frac{2}{x^2} - \frac{5}{x^3} - \frac{14}{x^4} - \frac{42}{x^5} - \frac{132}{x^7} - \cdots$$

$$= x - \sum_{n\geq 1} Cat_n \frac{1}{x^{n-1}}$$

$= x + b_1 + \frac{b_2}{x} + \frac{b_3}{x^2} + \cdots$

$= x - 1 - \sum_{n \geq 1} \frac{K_{n+1}^n}{n} \frac{1}{z^n}$.

This establishes that $-b_n(1,1,...,1) = Cat_n$.

Reduction to the Narayana polynomials (A001263)

The associahedra partition polynomials $ASP_n(c_1,...,c_n)$ are a refinement of the Narayana polynomials, and the following analysis will show that the partition polynomials $b_n(c_1,...,c_n) = b_n(a_1,...,a_n)$ are another refinement of the Narayana polynomials.

From A001263, a shifted o.g.f. $N(x,t)$ in x for the Narayana polynomials in t is

$$N(x,t) = \frac{1-\sqrt{1-\frac{4tx}{(1+(t-1)x)^2}}}{\frac{2}{1+(t-1)x}}$$

$$= tx + tx^2 + (t^2+t)x^3 + (t^3+3t^2+t)x^4 + (t^4+6t^3+6t^2+t)x^5$$

$$+ (t^5+10t^4+20t^3+10t^2+t)x^6 + \cdots$$

with the inverse about $x = 0$

$$N^{(-1)}(x,t) = \frac{x}{\frac{1}{1-x}-1+t} = \frac{x}{t+x+x^2+x^3+\cdots}.$$

For a specialization of our general formulas, let

$$f(x) = N^{(-1)}(x,t) = \frac{x}{\frac{1}{1-x}-1+t} = \frac{x}{t+x^2+x^3+\cdots} .$$

Then

$$f^{(-1)}(x) = N(x,t),$$

$$h(x) = \frac{x}{f(x)} = t + x + x^2 + x^3 + \cdots = \sum_{n \geq 0} c_n x^n ,$$

$$xg^{(-1)}(1/x) = \frac{x}{f^{(-1)}(x)} = \frac{x}{N(x,t)}$$

$$= \frac{\frac{2x}{1+(t-1)x}}{1-\sqrt{1-\frac{4tx}{(1+(t-1)x)^2}}} = \frac{1}{t}\frac{1+\sqrt{1-\frac{4tx}{(1+(t-1)x)^2}}}{\frac{2}{(1+(t-1)x)}}$$

$$= (\tfrac{1}{t})1 - (\tfrac{1}{t})x - (1)x^2 - (t+1)x^3 - (t^2+3t+1)x^4$$

$$- (t^3+6t^2+6t+1)x^5 - (t^4+10t^3+20t^2+10t-+1)x^6 + \cdots$$

$$= b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots,$$

$$g^{(-1)}(x) = \frac{1}{f^{(-1)}(1/x,\, t)} = \frac{1}{N(1/x,\, t)}$$

$$= \frac{x}{t}\frac{1+\sqrt{1-\frac{4\frac{t}{x}}{(1+\frac{t-1}{x})^2}}}{\frac{2}{(1+\frac{t-1}{x})}}$$

$$= (\tfrac{1}{t})x - (\tfrac{1}{t}) - \frac{1}{x} - \frac{t+1}{x^2} - \frac{t^2+3t+1}{x^3} - \frac{t^3+6t^2+61+1}{x^4} + \cdots$$

$$= b_0 x + b_1 + \frac{b_2}{x} + \frac{b_3}{x^2} + \cdots,$$ (expansion about $x = \pm\infty$),

and

$$g(x) = x\ h(1/x) = \frac{1}{f(1/x)} = \frac{1}{N^{(-1)}(1/x,t)}$$

$$= x\ (\frac{1}{1-\frac{1}{x}} - 1 + t)$$

$$= tx + 1 + \frac{1}{x} + \frac{1}{x^2} + \cdots$$

$$= c_0 x + c_1 + \frac{c_2}{x} + \frac{c_3}{x^2} + \cdots$$

This establishes that summing coefficients of the monomials in the numerators of the inversion polynomials $b_n(c_0, c_1, ..., c_n)$ with like powers k for c_0^k in the monomials generates the Narayana triangle.

Consistency check:

$$\frac{K_{n+1}^{n}}{n} = \frac{D_{x=0}^{5}}{5!}\ \frac{(h(x))^4}{4} = \frac{D_{x=0}^{5}}{5!}\,\frac{(t+x+x^2+x^3+x^4+x^5)^4}{4}$$

$$= \frac{D_{x=0}^{5}}{5!}[t^4/4 + t^3x + (t^3 + (3t^2)/2)x^2 + (t^3 + 3t^2 + t)x^3$$

$$+(t^3 + (9t^2)/2 + 3t + 1/4)x^4 + (t^3 + 6t^2 + 6t + 1)x^5 + \cdots]$$

$$= t^3 + 6t^2 + 6t + 1 = -b_5(c_0 = t, c_1 = 1, c_2 = 1, c_3 = 1, c_4 = 1, c_5 = 1).$$

The plots below, generated with the website Desmos with $t = 2.7$, provide a more complete, precise, picture of the inversions for the specialization for the Narayanas.

The curves plotted below in the four graphs are the

purple curve: bisecting diagonal, inverse curves are reflections through this diagonal

$y = x$

black curve: the function $y = g(x)$, with a singularity at $x = 1$

$$y = x\ (\tfrac{1}{1-x} - 1 + t)$$

$$= g(x) = x\ h(1/x) = \frac{1}{f(1/x)} = \frac{1}{N^{(-1)}(1/x,t)}$$

orange curve: inverse of the black curve $y = g(x)$, i.e., its reflection through $y = x$

$$x = y\ (\tfrac{1}{1-y} - 1 + t)$$

$$= g(y) = y\ h(1/y) = \frac{1}{f(1/y)} = \frac{1}{N^{(-1)}(1/y,t)}$$

red curve: inverse of $y = g(x)$ about $x - \pm\infty$ (and an interval near zero)

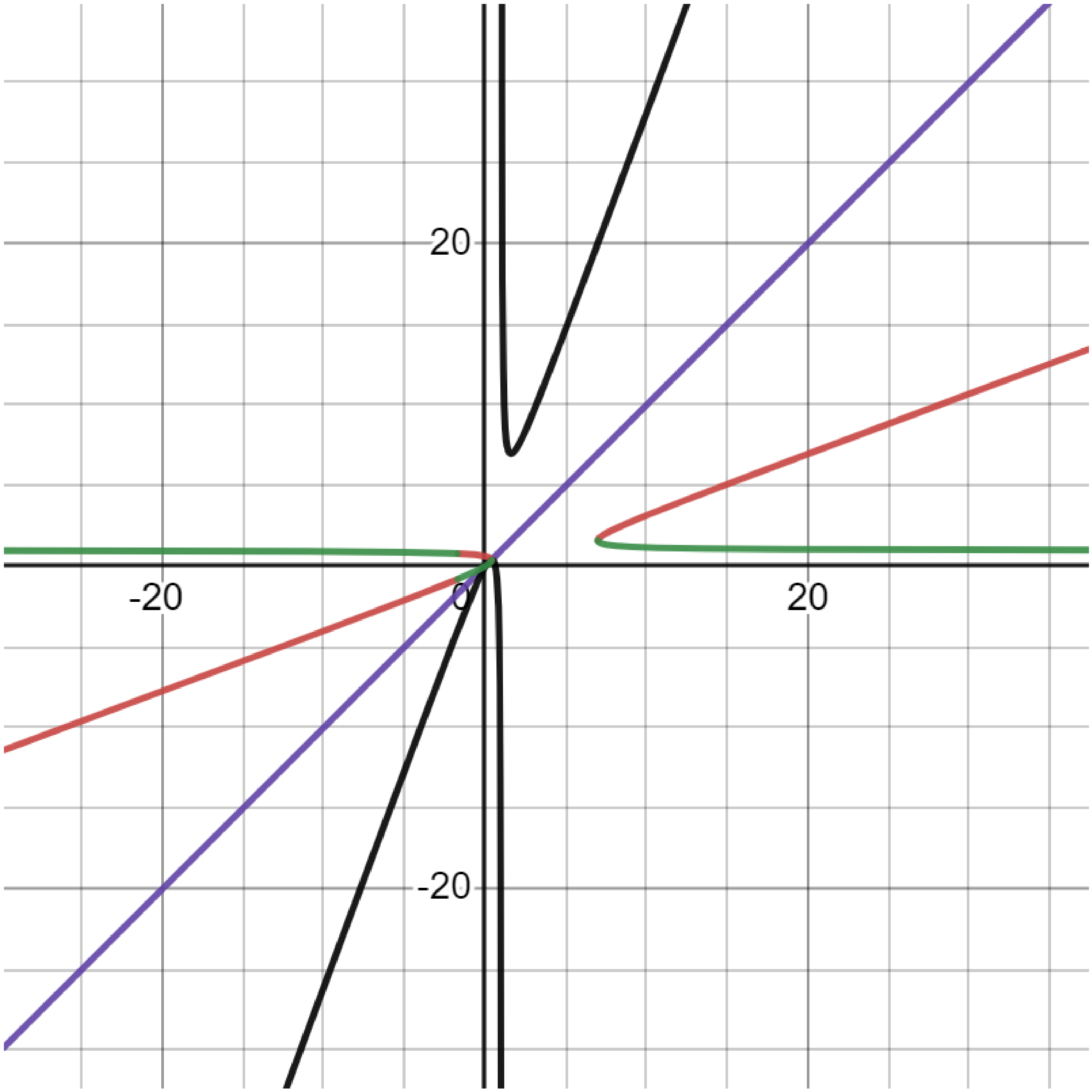
$$y = \frac{x}{t}\frac{1+\sqrt{1-\frac{4\frac{t}{x}}{(1+\frac{t-1}{x})^2}}}{\frac{2}{(1+\frac{t-1}{x})}}$$

$= g^{(-1)}(x) = \frac{1}{f^{(-1)}(1/x,\, t)} = \frac{1}{N(1/x,\, t)}$ when x gives a real y.

green curve: inverse of $y = g(x)$ for x in an interval about the origin

$$y = \frac{x}{t}\frac{1-\sqrt{1-\frac{4\frac{t}{x}}{(1+\frac{t-1}{x})^2}}}{\frac{2}{(1+\frac{t-1}{x})}}$$

$= g^{(-1)}(x) = \frac{1}{f^{(-1)}(1/x,\, t)} = \frac{1}{N(1/x,\, t)}$ when x gives a real y.



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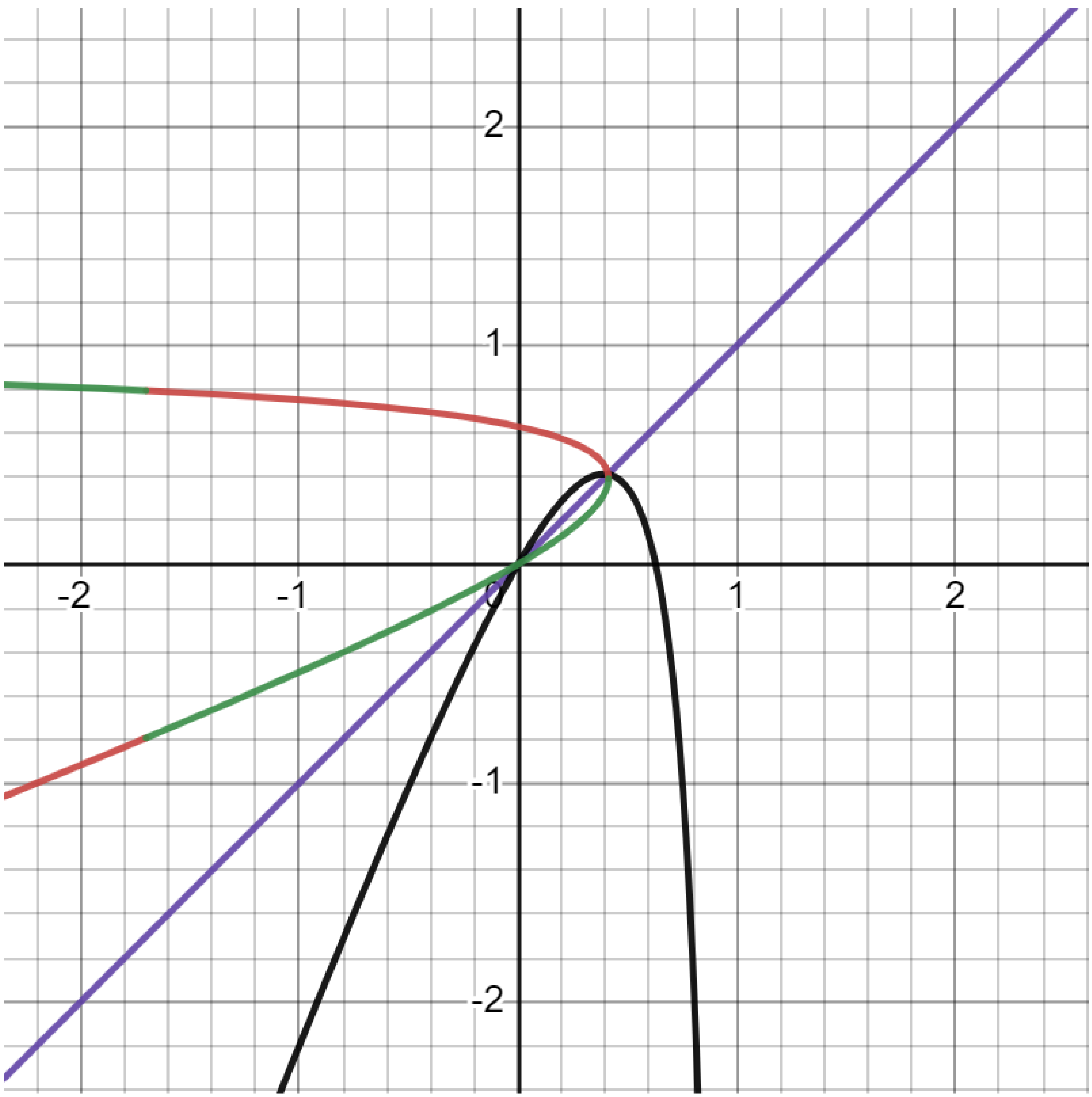

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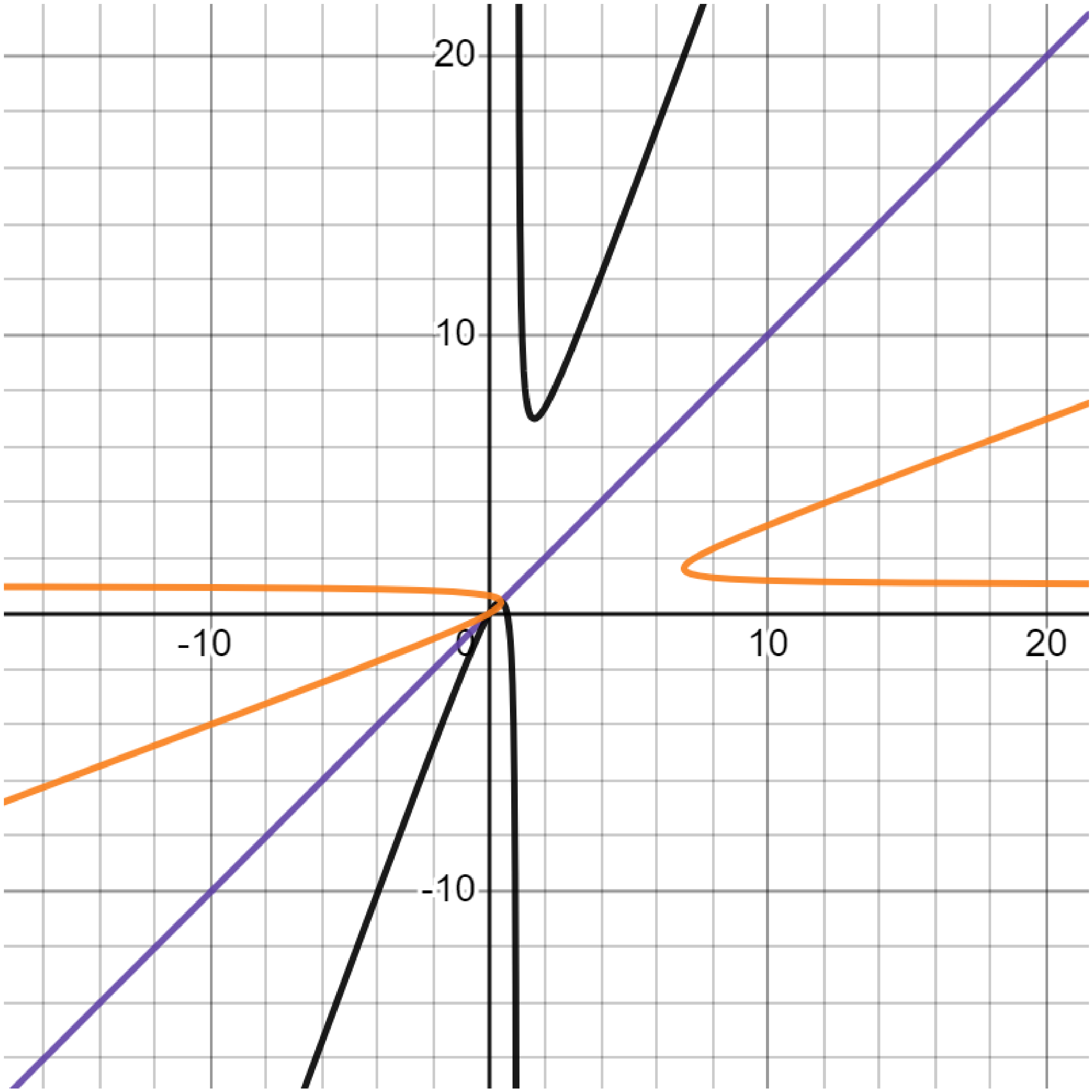
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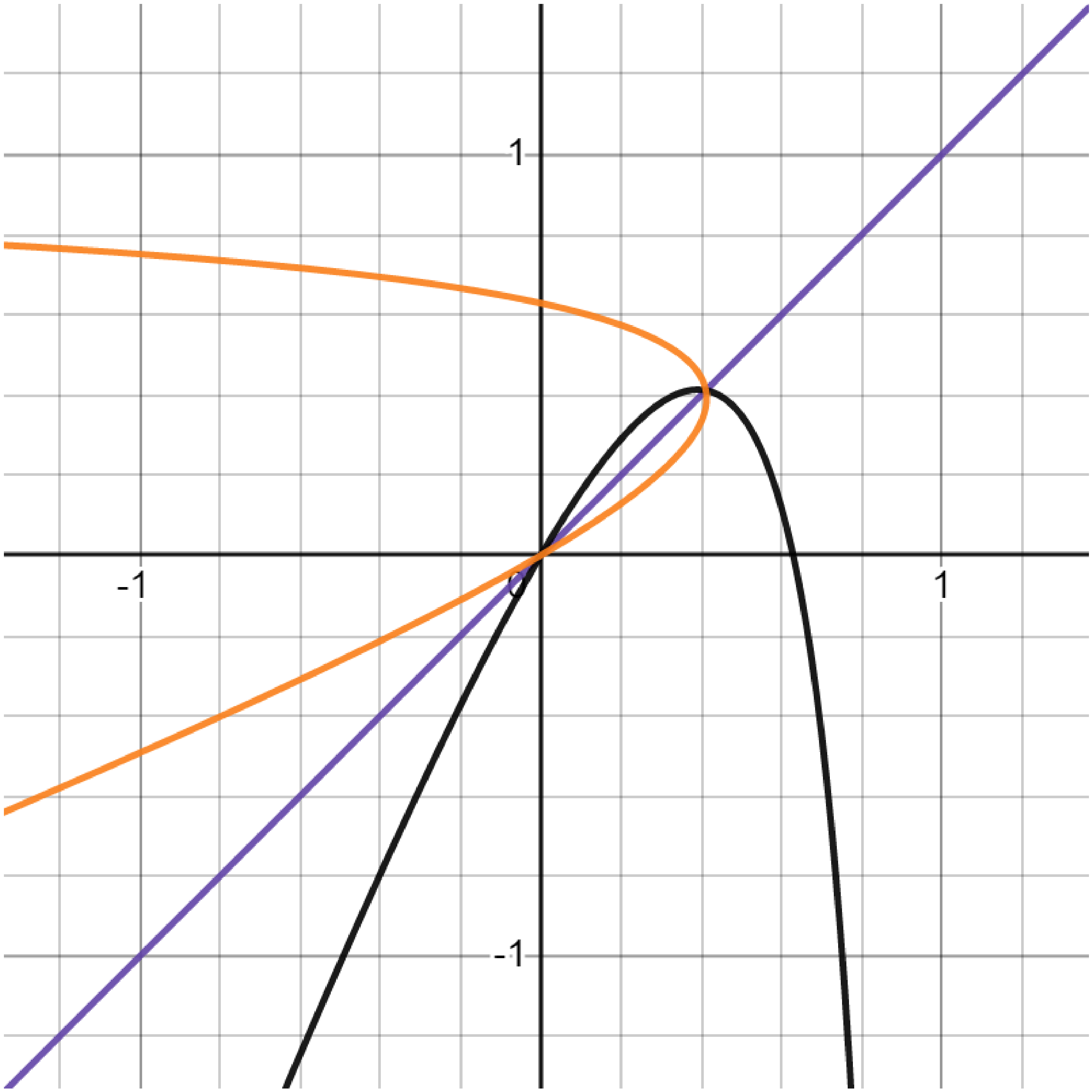

Reduction to the triangle A091869 enumerating Dyck paths and ordered trees

Above we found reductions of the coefficients of the inverse polynomials to the Catalans with $h(x) = 1 + x + x^2 + x^3 + \cdots$, i.e., with all $c_n = 1$, and to the Narayanas with $h(x) = t + x + x^2 + x^3 + \cdots$, i.e., with $c_0 = t$ and otherwise $c_n = 1$. Next consider

$h(x) = 1 + tx + x^2 + x^3 + x^4 + \cdots = \frac{x}{\frac{1}{1-x} - x + t}.$

For the triangle A091869, an o.g.f. Is

$$T(x,t) = \frac{(1+(1-t)x) - \sqrt{(1+(1-t)x)^2 - 4x(1+(1-t)x)}}{2}$$

$$= x + x^2 + (t+1)x^3 + (t^2+2t+2)x^4 + (t^3+3t^2+6t+4)x^5$$

$$+ (t^4 + 4t^3 + 12t^2 + 16t + 9)x^6 + \cdots ,$$

and these seem to be the polynomials of interest from reducing $b_n(c_1, ..., c_n)$ by substitution of $c_1 = t$ and $c_k = 1$ otherwise.

The shifted reciprocal is

$$\frac{x}{T(x,t)} = \frac{2x}{1+(1-t)x) - \sqrt{(1+(1-t)x)^2 - 4x(1+(1-t)x))}}$$

$$= 1 - x - tx^2 + (-t^2 - 1)x^3 + (-t^3 - 3t - 1)x^4 + (-t^4 - 6t^2 - 4t - 3)x^5 + \cdots .$$

For the second order terms and above, these are the signed polynomials of A091867, whose inverse is

$$\frac{x}{\frac{1}{1-x} - x + tx} = \frac{x}{1+tx+x^2+x^3+\cdots} = \frac{x-x^2}{1+(t-1)(x-x^2)},$$

a composition of the inverse $Cat^{(-1)}(x) = x - x^2$ of the shifted Catalans and a simply invertible Moebius transformation $\frac{z}{1+(t-1)z}$.

A091867 with an extra initial row has the o.g.f.

$$\frac{1-\sqrt{1-\frac{4x}{1+(1-t)x}}}{2} = x + tx^2 + (t^2+1)x^3 + (t^3+3t+1)x^4$$

$$+(t^4 + 6t^2 + 4t + 3)x^5 + (t^5 + 10t^3 + 10t^2 + 15t + 6)x^6 + \cdots$$

while its shifted reciprocal is

$$\frac{2x}{1-\sqrt{1-\frac{4x}{1+(1-t)x}}}$$

$$= 1 - tx - x^2 + (-t-1)x^3 + (-t^2-2t-2)x^4 + (-t^3-3t^2-6t-4)x^5 + \cdots,$$

an o.g.f. for the negated coefficients of A091869, which we wish to show are the $b_n(t,1,1,1,...,1)$, i.e., the sequence

$$1,\ -t, -1,\ -(t+1),\ -(t^2+2t+2),\ -(t^3+3t^2+6t+4), \cdots.$$

Now consider the Schur expansion coefficients.

$$\frac{(\frac{1}{1-x}-x+yx)^4}{4}$$
$$= 1/4 + yx + (\tfrac{3}{2}y^2+1)x^2 + (y^3+3y+1)x^3 + (\tfrac{1}{4}y^4+3y^2+3y+\tfrac{5}{2})x^4$$

$$+ (y^3+3y^2+6y+4)x^5 + \cdots,$$

the last polynomial is

$$\frac{K_5^4}{4} = (y^3+3y^2+6y+4),$$

the third order polynomials of A091869,

while the polynomial of the third power of x is

$$\frac{K_3^4}{4} = (y^3+3y+1),$$

the third order polynomial of A091867.

Likewise

$$\frac{(\frac{1}{1-x}-x+yx)^5}{5}$$

$$= \tfrac{1}{5} + yx + (2y^2+1)x^2 + (2y^3+4y+1)x^3 + (y^4+6y^2+4y+3)x^4$$

$$+(\tfrac{1}{5}y^5+4y^3+6y^2+10y+5)x^5 + (y^4+4y^3+12y^2+16y+9)x^6 + \cdots$$

with

$\frac{K_6^5}{5} = (y^4 + 4y^3 + 12y^2 + 16y + 9)$,

the fourth order polynomial of A091869,

and

$\frac{K_4^5}{5} = y^4 + 6y^2 + 4y + 3$,

the fourth order polynomial of A091867.

The highest exhibited coefficients, e.g., $(y^3 + 3y^2 + 6y + 4)$, are

$$\frac{K_{n+1}^n}{n} = \frac{D_{x=0}^{n+1}}{(n+1)!} \frac{(\frac{1}{1-x} - x + yx)^n}{n},$$

the polynomials of A091869. The third to the last coefficients, e.g., $(y^4 + 6y^2 + 4y + 3)$, are the expansion coefficients

$$\frac{K_{n-1}^n}{n} = \frac{D_{x=0}^{n-1}}{(n-1)!} \frac{(\frac{1}{1-x} - x + yx)^n}{n},$$

giving, via the Lagrange inversion formula, the coefficients of the compositional inverse of

$f(x) = \frac{x}{f(x)} = \frac{x}{\frac{1}{1-x} - x + yx}$, i.e., giving the polynomials of A091867.

Now with our just acquired familiarity with the relevant triangles, let's repeat the analysis for the Catalans to prove the alleged reduction/specialization for the special Schur coefficients with $c_1 = t$ and otherwise $c_n = 1$.

An o.g.f. for A091867 is

$$H(x,t) = \frac{1 - \sqrt{1 - \frac{4x}{1+(1-t)x}}}{2}$$

with the inverse in x about the origin

$$H^{(-1)}(x,t) = \frac{x}{\frac{1}{1-x} - x + tx} = \frac{x}{1 + tx + x^2 + x^3 + \cdots}$$

Let

$$f(x) = H^{(-1)}(x,t) = \frac{x}{\frac{1}{1-x} - x + tx}$$

Then

$$f^{(-1)}(x) = H(x,t)$$

and

$$h(x) = \frac{x}{f(x)} = \frac{x}{\frac{x}{\frac{1}{1-x} - x + tx}} = \frac{1}{1-x} - x + tx = 1 + tx + x^2 + x^3 + \cdots$$

$$= \sum_{n \geq 0} c_n x^n$$

Consequently, the general formulas give

$$g(x) = x\ h(1/x) = \frac{1}{f(1/x)} = x + c_1 + \frac{c_2}{x} + \frac{c_3}{x^2} + \cdots$$

$$= x + t + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots ,$$

and

$$xg^{(-1)}(1/x) = \frac{x}{f^{-1}(x)} = \frac{x}{H(x,t)} = \frac{2x}{1 - \sqrt{1 - \frac{4x}{1+(1-t)x}}}$$

$$= 2x \frac{1 + \sqrt{1 - \frac{4x}{1+(1-t)x}}}{\frac{4x}{1+(1-t)x}} = (1 + (1-t)x)\ \frac{1 + \sqrt{1 - \frac{4x}{1+(1-t)x}}}{2}$$

$$= \frac{1 + (1-t)x + \sqrt{(1+(1-t)x)^2 - 4x(1+(1-t)x)}}{2}$$

$$= (1 + (1-t)x) - \frac{(1+(1-t)x) - \sqrt{(1+(1-t)x)^2 - 4x(1+(1-t)x)}}{2}$$

$$= (1 + (1-t)x) - T(x,t)$$

$$= (1 + (1 - t)x) - [x + x^2 + (t+1)x^3 + (t^2 + 2t + 2)x^4 + (t^3 + 3t^2 + 6t + 4)x^5$$

$$+ (t^4 + 4t^3 + 12t^2 + 16t + 9)x^6 + \cdots]$$

$$= 1 - tx - x^2 - (t+1)x^3 - (t^2 + 2t + 2)x^4 - (t^3 + 3t^2 + 6t + 4)x^5$$

$$- (t^4 + 4t^3 + 12t^2 + 16t + 9)x^6 + \cdots]$$

$$= 1 - tx - \sum_{n \geq 0} T_n x^{n+2}.$$

T_n are the polynomials of A191869 with those of the infinite series just above, and these are the specialization for b_n. This claim follows from the general formulas as

$$xg^{(-1)}(1/x) = \frac{x}{f^{-1}(x)} = \frac{x}{H(x,t)}$$

$$= 1 - tx - \sum_{n \geq 0} T_n x$$

$$= 1 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots$$

$$= 1 + b_1 x - \sum_{n \geq 1} \frac{K_{n+1}^n}{n} \; x^{n+1}.$$

This establishes that $-b_n(t, 1, ..., 1) = T_n$ for $n > 1$ and $b_0 = 1$ and $b_1(c_1) = -c_1 = -t$.

Reduction again and mutual recursion formulas

Naturally, these reductions illustrate the following general relationships, a briefer presentation of the relationships used to prove the reduction results above. From these a recursion relation is evident.

Given

$$h(x) = 1 + c_1 x + c_2 x^2 + \cdots = \frac{x}{f(x)} = \frac{1}{1 + \hat{c}_1 x + \hat{c}_2 x^2 + \cdots},$$

$$b_{n+1}(c_1, ..., c_{n+1}) = -\frac{K_{n+1}^n(c_1, ..., c_{n+1})}{n} = -\frac{D_{x=0}^{n+1}}{(n+1)!} \frac{(h(x))^n}{n}$$

$$= -\frac{D_{x=0}^{n+1}}{(n+1)!} \frac{(\frac{x}{f(x)})^n}{n} = \frac{D_{x=0}^{n+1}}{(n+1)!} \frac{x}{f^{(-1)}(x)}$$,

which are the coefficients of the o.g.f. for $\frac{x}{f^{(-1)}(x)}$ with the obvious definitional convention $-\frac{K_{n+1}^n(c_1,...,c_{n+1})}{n}|_{n=0} := -c_1$,

whereas

$$NCP_{n-1}(c_1,...,c_{n-1}) = \frac{K_{n-1}^n(c_1,...,c_{n-1})}{n} = \frac{D_{x=0}^{n-1}}{(n-1)!} \frac{(h(x))^n}{n}$$

$$= \frac{D_{x=0}^{n-1}}{(n-1)!} \frac{(\frac{x}{f(x)})^n}{n} = \frac{D_{x=0}^{n}}{n!} f^{(-1)}(x)$$,

which are the coefficients of $f^{(-1)}(x)$ as a power series for $n \geq 1$.

Consequently, in the series expansion of $\frac{(h(x))^n}{n} = \frac{(\frac{x}{f(x)})^n}{n}$, we expect to see this relationship reflected in the polynomials of the $n-1$ and $n+1$ order coefficients of the expansion.

Then also since $\frac{f^{(-1)}(x)}{x} \frac{x}{f^{(-1)}(x)} = 1$, the two sets of polynomials $U_n(c_1,...,c_n)$ and $V_n(c_1,...,c_n)$ with the respective o.g.f.s $U(x) = \frac{f^{(-1)}(x)}{x}$ and $V(x) = \frac{x}{f^{(-1)}(x)}$ satisfy the convolution identity

$$\sum_{k=0}^n U_k V_{n-k} = \delta_n = 0^n$$.

Equivalently,

$$\sum_{k=0}^n \frac{K_k^{k+1}}{k+1} \frac{-K_{n-k}^{n-k-1}}{n-k-1} = \delta_n$$,

with, again, the convention $\frac{K_{n+1}^n}{n}|_{n=0} = c_1$,

and

$$\sum_{k=0}^n NCP_k(c_1,...,c_k)\; b_{n-k}(c_1,...,c_n) = \delta_n$$.

The convolution identity can be translated into a mutual pair of recursion relations (compare with the e.g.f. equivalent A133314): Given one sequence the other can be determined from

$U_n = -\sum_{k=0}^{n-1} U_k V_{n-k}$

or

$V_n = -\sum_{k=0}^{n-1} V_k U_{n-k}.$

Equivalently,

from

$\sum_{k=0}^{n} \frac{K_k^{k+1}}{k+1} \; \frac{-K_{n-k}^{n-k-1}}{n-k-1} = \delta_n$,

$\frac{K_n^{n+1}}{n+1} = -\sum_{k=0}^{n-1} \frac{K_k^{k+1}}{k+1} \; \frac{-K_{n-k}^{n-k-1}}{n-k-1}$

or

$\frac{-K_n^{n-1}}{n-1} = -\sum_{k=0}^{n-1} \frac{K_{n-k}^{n-k+1}}{n-k+1} \; \frac{-K_k^{k-1}}{k-1}.$

Equivalently,

$b_n(c_1, ..., c_n) = -\sum_{k=0}^{n-1} \frac{K_{n-k}^{n-k+1}}{n-k+1} \; b_k(c_1, ...c_k)$

$= -\sum_{k=0}^{n-1} NCP_{n-k}(c_1, ..., c_{n-k}) \; b_k(c_1, ...c_k)$

or

$\frac{K_n^{n+1}(c_1,...,c_n)}{n+1} = -\sum_{k=0}^{n-1} \frac{K_k^{k+1}}{k+1} \; b_{n-k}(c_1, ...c_{n-k})$

$= NCP_n(c_1, ..., c_n) = -\sum_{k=0}^{n-1} NCP_k(c_1, ..., c_k) \; b_{n-k}(c_1, ...c_{n-k}).$

Example:

From the last reduction with $h(x) = \frac{1}{1-x} - x + tx = 1 + tx + x^2 + x^3 + \cdots$,

$$f(x) = \frac{x}{h(x)} = \frac{x}{1+tx+x^2+x^3+\cdots} = \frac{x}{\frac{1}{1-x}-x+tx} = \frac{x(1-x)}{1+(t-1)x(1-x)} = \frac{x-x^2}{1+(t-1)(x-x^2)}$$

$= \frac{z}{1+(t-1)z}$ with $z = x(1-x) = x - x^2$.

The inverse of the linear fractional, or Moebius, transformation

$$M(z) = \frac{z}{1+(t-1)z}$$

is

$$M^{(-1)}(z) = \frac{z}{1-(t-1)z}$$

as is easily determined from the inverse of the matrix $\begin{pmatrix} 1 & 0 \\ (t-1) & 1 \end{pmatrix}$, which is $\begin{pmatrix} 1 & 0 \\ (1-t) & 1 \end{pmatrix}$.

The inverse of the Catalan sequence o.g.f.

$$Cat(x) = \frac{1-\sqrt{1-4x}}{2} = x + x^2 + 2x^3 + 5x^4 + 14x^5 + 42x^6 + \cdots$$

is

$$Cat^{(-1)}(x) = x(1-x) = x - x^2.$$

Then

$$f(x) = M[Cat^{(-1)}(x)] = \frac{x-x^2}{1+(t-1)(x-x^2)} = \frac{x}{\frac{1}{1-x}-x+tx}$$

$$= \frac{x(1-x)}{1+(t-1)x(1-x)} = \frac{x}{1+tx+x^2+x^3+\cdots} = x/h(x)$$

with inverse in x about the origin

$$f^{(-1)}(x) = Cat[M^{(-1)}(x)] = \frac{1-\sqrt{1-4(\frac{x}{1-(t-1)x})}}{2}$$

$$= x + tx^2 + (t^2+1)x^3 + (t^3+3t+1)x^4 + (t^4+6t^2+4t+3)x^5$$

$$+ (t^5 + 10t^3 + 10t^2 + 15t + 6)x^6 + \cdots$$

which is an o.g.f. for A091867.

Then

$$U(x) = \frac{f^{(-1)}(x)}{x} = \frac{Cat[M^{(-1)}(x)]}{x}$$

$$= 1 + tx + (t^2+1)x^2 + (t^3+3t+1)x^3 + (t^4+6t^2+4t+3)x^4$$

$$+(t^5 + 10t^3 + 10t^2 + 15t + 6)x^5 + \cdots$$

and

$$V(x) = \frac{x}{f^{(-1)}(x)} = \frac{x}{Cat[M^{(-1)}(x)]}$$

$$= 1 - tx - x^2 - (t+1)x^3 - (t^2+2t+2)x^4 - (t^3+3t^2+6t+4)x^5 + \cdots .$$

Since

$$\frac{x}{Cat(x)} = \frac{2x}{1-\sqrt{1-4x}} = 2x\,\frac{1+\sqrt{1-4x}}{4x} = \frac{1+\sqrt{1-4x}}{2},$$

$$\frac{x}{Cat[M^{(-1)}(x)]} = x\,\frac{1+\sqrt{1-4M^{(-1)}(x)}}{2M^{(-1)}(x)}$$

$$= x\,\frac{1+\sqrt{1-4(\frac{x}{1+(1-t)z})}}{2(\frac{x}{1+(1-t)z})}$$

$$= \frac{1+(1-t)z+\sqrt{(1+(1-t)z)^2-4x(1+(1-t)z}}{2}$$

$$= 1+(1-t)z - \frac{1+(1-t)z-\sqrt{(1+(1-t)z)^2-4x(1+(1-t)z}}{2}$$

$$= 1+(1-t)z - T(x,t)$$

where $T(x,t)$ is an o.g.f. for A091869 as can be confirmed by solving the quadratic formula in that entry $z\,G^2 - (1+z-tz)G + 1 + z - tz$ for G. This all agrees with the earlier analysis for the reduction.

The results could easily be extended to the Fuss-Catalan sequences, whose o.g.f.s $FC_n(x)$ are the inverse of $FC_n^{(-1)}(x) = x - x^{n+1}$. The Fuss-Catalans have several combinatorial interpretations and would provide other simple specializations of b_n with potentially useful interpretations.

Now let's check that the Cauchy convolution identity holds with our example polynomials.

$\sum_{k=0}^{0} U_k V_{n-k} = U_0 V_0 = 1 \cdot 1 = 1$,

$\sum_{k=0}^{1} U_k V_{n-k} = U_1 V_0 + U_0 V_1 = t - t = 0$,

$\sum_{k=0}^{2} U_k V_{n-k} = U_2 V_0 + U_1 V_1 + U_0 V_2 = (t^2+1) - t^2 + (-1) = 0$,

$\sum_{k=0}^{3} U_k V_{n-k} = U_3 V_0 + U_2 V_1 + U_1 V_2 + U_0 V_3$

$= (t^3 + 3t + 1) + (t^2+1)(-t) + (t)(-1) + (-(t+1)) = 0$.

Check that the indices are correct for the convolution of the K_n^p.

$U_3 = \frac{D_{x=0}^3}{3!} U(x) = \frac{D_{x=0}^3}{3!} \frac{f^{(-1)}(x)}{x} = \frac{D_{x=0}^4}{4!} f^{(-1)}(x) = \frac{K_3^4}{4} = \frac{D_{x=0}^3}{3!} \frac{(h(x))^4}{4}$

$= \frac{D_{x=0}^3}{3!} \frac{(\frac{x}{f(x)})^4}{4} = \frac{D_{x=0}^3}{3!} \frac{(\frac{1}{1-x} - x + tx)^4}{4} = t^3 + 3t + 1$.

$V_3 = \frac{D_{x=0}^3}{3!} V(x) = \frac{D_{x=0}^3}{3!} \frac{x}{f^{(-1)}(x)} = \frac{-K_3^2}{2} = -\frac{D_{x=0}^3}{3!} \frac{(h(x))^2}{2} = -\frac{D_{x=0}^3}{3!} \frac{(\frac{x}{f(x)})^2}{2}$

$= -\frac{D_{x=0}^3}{3!} \frac{(\frac{1}{1-x} - x + tx)^2}{2} = -(t+1)$.

From the convolution identity,

$\sum_{k=0}^{n} \frac{K_k^{k+1}}{k+1} \frac{-K_{n-k}^{n-k-1}}{n-k-1} = \delta_n$,

$\frac{K_0^1}{1} \frac{-K_0^{-1}}{-1} = (1)(1 = 1$,

$\sum_{k=0}^{1} \frac{K_k^{k+1}}{k+1} \frac{-K_{n-k}^{n-k-1}}{n-k-1} = \frac{K_0^1}{1} \frac{-K_1^0}{0} + \frac{K_1^2}{2} \frac{-K_0^{-1}}{-1} = (1)(= t) + (t) = 0$,

$\sum_{k=0}^{2} \frac{K_k^{k+1}}{k+1} \frac{-K_{n-k}^{n-k-1}}{n-k-1} = \frac{K_0^1}{1} \frac{-K_2^1}{1} + \frac{K_1^2}{2} \frac{-K_1^0}{0} + \frac{K_2^3}{3} \frac{-K_0^{-1}}{-1}$

$= (1)(-1) + (t)(-t) + (t^2+1)(1) = 0$.

A check of the recursion formula:

$NCP_n(c_1, ..., c_n) = -\sum_{k=0}^{n-1} NCP_k(c_1, ..., c_k)\ b_{n-k}(c_1, ...c_{n-k})$,

in particular

$NCP_3(c_1, ..., c_3) = -\sum_{k=0}^{2} NCP_k(c_1, ..., c_k)\ b_{3-k}(c_1, ...c_{3-k})$

$= -[NCP_0\ b_3 + NCP_1\ b_2 + NCP_2\ b_1]$

$= -[-(c_1c_2 + c_3) + (c_1)(-c_2) + (c_2 + c_1^2)(-c_1)]$

$= c_3 + 3c_1c_2 + c_1^3$.

Conversely,

$b_3 = -\sum_{k=0}^{2} b_k\ NCP_{3-k} = -[b_0 NCP_3 + b_1 NCP_2 + b_2 NCP_1]$

$= -[(c_3 + 3c_1c_2 + c_1^3) + (-c_1)(c_1^2 + c_2) + (-c_2)(c_1)] = -(c_3 + c_1c_2)$.

Example of the involutive property of the special Schur self-Konvolution partition polynomials

The set of involutive Schur self-Konvolution partition polynomials is one of three examples presented in my MathOverflow question “Examples of infinite dimensional involutions”.

For example, for the first four grades of involution are,

$b_0 = \frac{1}{a_0} = \frac{1}{\frac{1}{b_0}} = b_0$

$b_1 = -\frac{a_1}{a_0} = -\frac{-(b_1/b_0)}{1/b_0} = b_1$

$b_2 = -a_2 = -(-b_2) = b_2$

$b_3 = -(a_1 a_2 + a_0 a_3) = -[-(\frac{b_1}{b_0})(-b_2) + \frac{1}{b_0}((-(b_1 b_2 + b_0 b_3))] = b_3.$

Note that the partition a_1^n doesn't occur in b_n for $n > 1$.

Convolutions and umbral calculus:

Convolutions and umbral calculus:

The Schur coefficients can be calculated by an algebraic umbral method described in my post "The Hirzebruch criterion for the Todd class" and applied to A134264. Switch from an o.g.f. formulation to an e.g.f. by letting $c_n = a_n = q_n/n!$. Then, in umbral notation, e.g., with $(a.)^k = a_k$, and with $(a.)^0 = a_0 = c_0 = q_0 = 1$,

$$h(x) = \sum_{n \geq 0} c_n x^n = e^{q.x}$$

and

$$(h(x))^n = (\langle e^{q.x} \rangle)^n,$$

where $\langle \cdots \rangle$ indicates that the umbral evaluation is to be made before exponentiation by n. This is also equal to

$$(h(x))^n = e^{q^{(1)}.x} e^{q^{(2)}.x} \cdots e^{q^{(n)}.x} = \exp[(q.^{(1)} + q.^{(2)} + \cdots + q.^{(n)})x] \,|_{(q.^{(k)})^m \to q_m} \quad ,$$

for which, after expansion in a Taylor series in x and reduction of all the polynomials to monomial summands such as $N \; (q^{(5)}.)^4 (q^{(2)}.)^1$, where N is some natural number, the superscripts in parentheses are ignored and the power dropped to the empty subscript position marked by the dots, or periods.

As an illustration and check of the computation, consider

$$-b_4 = \frac{K_4^3}{3} = \frac{D_{x=0}^4}{4!}\frac{(h(x))^3}{3} = \frac{D_{x=0}^4}{4!}\frac{(\langle e^{q.x}\rangle)^3}{3}$$

$$= \frac{D_{x=0}^4}{4!}\frac{\exp[(q.^{(1)}+q.^{(2)}+q.^{(3)})\ x]}{3}\ \Big|_{(q.^{(k)})^m \to\ q_m}$$

$$= \frac{(q.^{(1)}+q.^{(2)}+q.^{(3)})^4}{4!\ 3}\ \Big|_{(q.^{(k)})^m \to\ q_m} \quad .$$

The numerator can be re-expressed to reduce the clutter yet still stress the distinctions among the umbral variables as

$$(q.^{(1)} + q.^{(2)} + q.^{(3)})^4 = (A. + B. + C.)^4 = (U_1 + U_2 + U_3)\ ,$$

where in the last expression the flag, or marker, for an umbral quantity, i.e., the dot or period, is omitted.

Then use your preferred symbolic math app to obtain

$$(U_1 + U_2 + U_3)^4 = U_1^4 + 4U_2U_1^3 + 4U_3U_1^3 + 6U_2^2U_1^2 + 6U_3^2U_1^2$$

$$+ 12U_2U_3U_1^2 + 4U_2^3U_1 + 4U_3^3U_1 + 12U_2U_3^2U_1 + 12U_2^2U_3U_1 + U_2^4 + U_3^4$$

$$+ 4U_2U_3^3 + 6U_2^2U_3^2 + 4U_2^3U_3$$

with 15 terms and the coefficients summing to $(1+1+1)^4 = 3^4 = 81$, the number of ways three symbols can be permuted with replacement in four linear positions. The coefficient of each monomial $U_m^j\ U_n^k$ is $\frac{(j+k)!}{j!\ k!} = \binom{j+k}{j,\ k} = \binom{4}{j,\ k}$.

Return to the meaning of the umbral variables $U_1,\ U_2$ and U_3. As constructed, they are not independent rather $U_1^m = U_2^m = U_3^m = q_m$ in the final unique expansion, so now perform umbral reduction in the last expansion and accumulation of umbral variables by erasing subscripts and then dropping/lowering the powers into the subscript positions to obtain

$$(U_1 + U_2 + U_3)^4 \to\ q_4 + 4q_1q_3 + 4q_1q_3 + 6q_2q_2 + 6q_2q_2 + 12q_1q_1q_2$$

$$+ 4q_3q_1 + 4q_3q_1 + 12q_1q_2q_1 + 12q_2q_1q_1 + q_4 + q_4 + 4q_1q_3 + 6q_2q_2 + 4q_3q_1$$

$$= (1+1+1)q_4 + (4+4+4+4+4+4)q_1q_3 + (6+6+6)q_2^2 + (12+12+12)q_1^2q_2$$

$= 3 \cdot 1q_4 + 6 \cdot 4q_1q_3 + 3 \cdot 6q_2^2 + 3 \cdot 12q_1^2q_2$

$= 3q_4 + 24q_1q_3 + 18q_2^2 + 36q_1^2q_2.$

As a quick check note that these are integer partitions of 4, i.e., all the subscripts sum to 4 for each monomial. The coefficients enumerate the number of ways that 3 symbols can be permuted in 4 linear positions (e.g., a list, or a word, four letters long) with replacement such that the number of times a symbol occurs in a list is designated by the subscript. For example, only 3 distinct words can be formed from the three letters R, S, and T--(R,R,R,R), (S,S,S,S), and (T,T,T,T)--in which the same symbol occurs 4 times; hence, the term $3q_4$. For the monomial q_1q_3 each word has only two distinct symbols with one symbol occurring once and the other thrice. Note the monomial q_1^4 does not appear since this would require the initial expansion to contain a term such as $q_jq_kq_mq_n$ with $j+k+m+n=4$ such that j,k,m,n are all distinct numbers and only three distinct numbers are available.

Converting to the coefficients of the original o.g.f. gives

$(U_1 + U_2 + U_3)^4 \rightarrow\ 3q_4 + 24q_1q_3 + 18q_2^2 + 36q_1^2q_2 = 3 \cdot 4!c_4 + 24 \cdot 3!c_1c_3$

$+ 18 \cdot 2^2c_2^2 + 36 \cdot 2c_1^2c_2$

$= 72c_4 + 144c_1c_3 + 72c_2^2 + 72c_1^2c_2.$

Finally dividing by $((n+1)! \cdot n)\ |_{n=3} = 4! \cdot 3 = 72$ gives, correctly,

$-b_4 = \frac{K_4^3}{3} = c_4 + 2c_1c_3 + c_2^2 + c_1^2c_2.$

The expression $(A_1 + A_2 + A_3)^4 = (U_1 + U_2 + U_3)^4$ is a symmetric homogeneous polynomial in three variables/indeterminates, and the coefficients of the monomials of the expansion enumerate the various permutations of these three symbols with replacement.

Basic treatises on algebraic combinatorics state that, given distinct objects C_m of size /cardinality c_m, the coefficient of x^k in the expansion of $(h(x))^n = (1 + c_1x + c_2x^2 + \cdots)^n$ counts sequences of objects $(C_1, C_2, ..., C_n)$ with a total size over all objects of k.

Other illustrations / spot checks:

$(U_1+U_2)^3 = U_1^3 + 3U_2U_1^2 + 3U_2^2U_1 + U_2^3.$

Erase subscripts and lower powers, change variables, and then accumulate the monomials:

$U_1^3 + 3U_2U_1^2 + 3U_2^2U_1 + U_2^3 \to U3 + 3U_1U_2 + 3U_2U_1 + U_3$

$= q_3 + 3q_1q_2 + 3q_2q_1 + q_3 = (1+1)q_3 + (3+3)q_1q_2 = 2q_3 + 6q_1q_2.$

Change variables ($q_k = k!c_n = k!a_n$) again:

$2q_3 + 6q_1q_2 = 2\cdot 3!c_3 + 6\cdot 2c_1c_2 = 12c_3 + 12c_1c_2$.

Divide by $3!\cdot 2 = 12$:

$-b_3 = \frac{K_3^2}{2} = c_3 + c_1c_2$.

Remark: the number of distinct monomials in the expansion of $(U_1+\cdots+U_n)^{n+1}$ is given by https://oeis.org/A001791 = (0, 1, 4, 15, 56, 210, …) = $\binom{2n}{n-1}$.

Multinomial expansion:

First,

$(A_1+B)^{n+1} = \sum_{j=0}^{n+1}\binom{n+1}{j}A_1^{n+1-j}B^j$.

Then with

$B = A_2 + C$,

$(A_1+A_2+C)^{n+1} = \sum_{j=0}^{n+1}\binom{n+1}{j}A_1^{n+1-j}(A_2+C)^j$

$= \sum_{j=0}^{n+1}\binom{n+1}{j}A_1^{n+1-j}\sum_{k=0}^{j}\binom{j}{k}A_2^{k-j}C^j$, and so on so that with $n-1$ summation symbols

$(U_1+U_2+...+U_n)^{n+1}$

$= \sum_{k_1=0}^{n+1} \binom{n+1}{k_1} U_1^{n+1-k_1} [\sum_{k_2=0}^{k_1} \binom{k_1}{k_2} U_2^{k_1-k_2} \; [\sum_{k_3=0}^{k_2} \binom{k_2}{k_3} U_3^{k_2-k_3}$

$[...[\sum_{k_{n-1}=0}^{k_{n-2}} \binom{k_{n-2}}{k_{n-1}} U_{n-1}^{k_{n-2}2-k_{n-1}}]]]...]$

$= \sum_{k_1+k_2+...+k_n=n} \frac{(n+1)!}{k_1!,k_2!,...,k_n!} U_1^{k_1} U_2^{k_2} ... U_n^{k_n}$

$= \sum_{k_1+k_2+...+k_n=n} \binom{n+1}{k_1,k_2,...,k_n} U_1^{k_1} U_2^{k_2} ... U_n^{k_n}$.

Examples:

$n=2$:

$n-1=1$, so we have only one summation symbol

$(U_1+U_2)^3 = U_1^3 + 3U_2U_1^2 + 3U_2^2U_1 + U_2^3$

(4 terms, sum of coefficients $= 2^3 = 8$)

$(U_1+U_2)^3 = \sum_{k_1=0}^{3} \binom{3}{k_1} U_1^{3-k_1} U_2^{k_1}$

$= U_1^3 + 3U_2U_1^2 + 3U_2^2U_1 + U_2^3$.

For $n=3$:

$n-1=2$, so we have the product of two summation symbols

$(U_1+U_2+U_3)^4 = U_1^4 + 4U_2U_1^3 + 4U_3U_1^3 + 6U_2^2U_1^2 + 6U_3^2U_1^2$

$$+ 12U_2U_3U_1^2 + 4U_2^3U_1 + 4U_3^3U_1 + 12U_2U_3^2U_1 + 12U_2^2U_3U_1 + U_2^4$$

$$+ U_3^4 + 4U_2U_3^3 + 6U_2^2U_3^2 + 4U_2^3U_3$$

(15 terms, coefficients sum to $3^4 = 81$)

$$(U_1 + U_2 + U_3)^4 = \sum_{k_1=0}^{4} \sum_{k_2=0}^{k_1} \frac{4!}{(4-k_1)!(k_1-k_2)!k_2!} \, U_1^{4-k_1} U_2^{k_1-k_2} U_3^{k_2}$$

1) $k_1 = 0$ allows only $k_2 = 0$ giving U_1^4.

2) $k_1 = 1$ and $k_2 = 0$ gives $(\frac{4!}{3!} = 4)U_1^3U_2$

3) $k_1 = 1$ and $k_2 = 1$ gives $(\frac{4!}{3!} = 4)U_1^3U_3$

4) $k_1 = 2$ and $k_2 = 0$ gives $(\frac{4!}{2!2!} = 6)U_1^2U_2^2$

5) $k_1 = 2$ and $k_2 = 1$ gives $(\frac{4!}{2!} = 12)U_1^2U_2U_3$.

Because the polynomial are symmetric under permutations of the symbols, we have only four unique coefficients and patterns that remain the same if different symbols are associated with the subscripted ones: $U_1^4 \to A^4$, $4U_1^3U_2 \to 4A^3B$, $6U_1^2U_2^2 \to 6A^2B^2$ and $12U_1^2U_2U_3 \ \to 12A^2BC$.

Then, from the symmetry, there should be 3 terms of type A^4, i.e., the forms A^4, B^4, C^4; $3 \cdot 2 = 6$ terms of type $4A^3B$, i.e., the combinations (without replacement) A^3B, A^3C, B^3A, B^3C, C^3A, C^3B; $(3 \cdot 2)/2 = 3$ terms of type $6A^2B^2$, i.e., A^2B^2, A^2C^2, and B^2C^2; and $\frac{3!}{2!} = 3$ combinations (without replacement) of type $12A^2BC$, i.e., A^2BC, B^2AC, and C^2AB.

The umbrally reduced monomials are $A^4 \to q_4$, $A^3B \to q_3q_1$, $A^2B^2 \to q_2q_2 = q_2^2$, and $A^2BC \to q_2q_1q_1 = q_2q_1^2$.

Then reduction with the umbral power lowering maneuver $U_k^m \ \to q_m$ gives

$$(U_1 + U_2 + U_3)^4 \to 3q_4 + 6 \cdot 4q_3q_1 + 3 \cdot 6q_2^2 + 3 \cdot 12q_2q_1^2 = 3q_4 + 24q_3q_1 + 18q_2^2 + 36q_2q_1^2.$$

Converting with $q_k = k!c_k$, this becomes

$$3 \cdot 4!c_4 + 6 \cdot 4 \cdot 3!c_3c_1 + 3 \cdot 6 \cdot 2^2c_2^2 + 3 \cdot 12 \cdot 2c_2c_1^2$$

$$= 72c_4 + 144c_3c_1 + 72c_2^2 + 72c_2c_1^2.$$

Finally, dividing by $(n+1)!\ n = 4!\ 3 = 72$ gives

$$-b_4 = c_4 + 2c_3c_1 + c_2^2 + c_2c_1^2.$$

One could think of the computations as successive decreasing levels of refinement or increasing levels of coarseness in terms of sentences containing words:

The most refined level corresponds to the expansion of $(A_1 + A_2 + A_3)(B_1 + B_2 + B_3)(C_1 + C_2 + C_3)(D_1 + D_2 + D_3)$ treating the symbols as noncommutative, generating $3^4 = 81$ sentences with four words each and each word containing only one letter.

Next, the words containing the same letter in a sentence are merged into a single larger word in a sentence neglecting word order so that each word in a new sentence now contains a letter distinct from those of the other words and possibly a different number of letters from the other words in that sentence, that number being the word length. This corresponds to the expansion of $(U_1 + U_2 + U_3)^4$, accumulating like monomials treating each letter/symbol/indeterminate as commutative. The power of the indeterminate is the length of the corresponding word. Each monomial is a grammarless sentence, i.e., the word order is insignificant, with a total length in letters of four. The total sum of the numerical coefficients multiplying these monomials/sentences is $3^4 = 81$, and the numerical coefficient multiplying a monomial/grammarless sentence is the multinomial coefficient.

The next level of coarsening is to ignore the letters in a word and count the number of sentences with the same number of words and word lengths. This corresponds to erasing the subscript of our indeterminates and dropping, or lowering, the power into the empty subscript position; e.g. $U_k^m \to U_m \to q_m$. Consolidating, or accumulating, the reduced symbols gives a monomial of the form $q_j^k\ q_m^p$ with the sum of the products of subscripts with the corresponding powers now four for each monomial, i.e., $k \cdot j + p \cdot m = 4$.

Generalizing to any $n > 1$, we have then $n+1$ indeterminates q_k comprising the monomials of $-b_{n+1}$ and K_{n+1}^n of the form

$$MBQ_{n+1}[e_1, e_2, ..., e_{n+1}] = q_1^{e_1} q_2^{e_2} ... q_{n+1}^{e_{n+1}}.$$

Each factor $q_k^{e_k}$ in the monomial for which $e_k > 0$ can be associated with a proliferation of symbols, or letters: $q_1^{e_1}$ to a product of e_1 distinct letters, $q_2^{e_2}$ to a product of e_2 doubled letters, $q_3^{e_3}$ to a product of e_3 tripled letters, and so on with each multiplet of letters/symbols being distinct. For example, q_1^3 can be associated with $ABC = L_1^1 L_2^1 L_3^1 = S_{1,1}^1 S_{1,2}^1 S_{1,3}^1$, which umbrally reduces to $q_1 q_1 q_1 = q_1^3$; and q_2^4 can be associated with $D^2 E^2 F^2 G^2 = L_4^2 L_5^2 L_6^2 L_7^2 = S_{2,1}^2 S_{2,2}^2 S_{2,3}^2 S_{2,4}^2$, which umbrally reduces to $q_2 q_2 q_2 q_2 = q_2^4$. In general, the factor $q_k^{e_k}$ can be associated with the symbol form $\prod_{m=1}^{e_k} S_{k,m}^k$ with all symbols $S_{k,m}$ distinct for distinct pairs of natural numbers (k,m) with $k, m > 0$.

The symbol form associated with the monomial $MBQ_{n+1}[e_1, e_2, ..., e_{n+1}]$ is

$$SM_{n+1}[e_1, e_2, ..., e_{n+1}]$$

$$= (S_{1,1}^{\bar{e}_{1,1}} S_{1,2}^{\bar{e}_{1,2}} \cdots S_{1,e_1}^{\bar{e}_{1,e_1}})\,(S_{2,1}^{\bar{e}_{2,1}} S_{2,2}^{\bar{e}_{2,2}} \cdots S_{2,e_2}^{\bar{e}_{2,e_2}}) \cdots (S_{n+1,1}^{\bar{e}_{n+1,1}} S_{n+1,2}^{\bar{e}_{n+1,2}} \cdots S_{n+1,e_{n+1}}^{\bar{e}_{n+1,e_{n+1}}}).$$

All these considerations ignore the $q_0 = a_0 = c_0$ indeterminates--they can be regarded as equal to unity and do not enter into calculations of the coefficients of the monomials. Note that $\bar{e}_{k,m} = k$ for $0 \leq m \leq e_k$ and that e_k distinct letters are associated with each q_k and, therefore, c_k or a_k in the associated monomials. This applies only to those q_k whose exponents e_k are not zero, i.e., those q_k which naturally appear in the monomial for $k > 0$. Stressing once more, upon umbral evaluation, the symbols associated with a q_k lose their individuality and are then multiplied together (or coalesce) to give $q_k^{e_k}$.

From the combinatorics of the expansion of $(U_1 + U_2 + \cdots + U_n)^{n+1}$ and subsequent umbral reduction and accumulation of symbol forms, with the relabeling

$$(E_1, E_2, E_3, ..., E_{NZE}) = (\bar{e}_{1,1}, \bar{e}_{1,2}, ..., \bar{e}_{1,e_1}, \bar{e}_{2,1}, ..., \bar{e}_{1,e_1}, ...),$$

with $NZE = e_1 + e_2 + \cdots + e_{n+1}$ being the number of non-zero exponents $\bar{e}_{k,m}$,

the coefficient of the symbol form is

$$\frac{(n)_{for\ first\ E_k}(n-1)_{for\ second\ E_k}\cdots}{(no.\ of\ E_k\ equal\ to\ 1)!\ (no.\ of\ E_k\ equal\ to\ 2)!...(no.\ of\ E_k\ equal\ to\ n+1)!}\ \frac{(n+1)!}{E_1!E_2!...E_{NZE}!}$$

$$= \frac{\frac{n!}{(n-NZE)!}}{(no.\ of\ E_k\ equal\ to\ 1)!\ (no.\ of\ E_k\ equal\ to\ 2)!...(no.\ of\ E_k\ equal\ to\ n{+}1)!}\ \frac{(n+1)!}{E_1!E_2!...E_{NZE}!}.$$

Check with $n = 4$:

$$(U_1 + U_2 + U_3 + U_4)^5 \ \rightarrow 4q_5 + 60q_1q_4 + 120q_2q_3 + 240q_1^2q_3 + 360q_1q_2^2 + 240q_1^3q_2.$$

(DIvide by 4 and reverse reorder to obtain 60,90,60,30,20,1 in A & S order. Neither vector is in the OEIS.)

The monomial q_5 comes from the form $A^5 = A^{E_1}$

with $E_1 = 5$, so the coefficient is

$$\frac{(n)_{for\ first\ E_k}(n-1)_{for\ second\ E_k}\cdots}{(no.\ of\ E_k\ equal\ to\ 1)!\ (no.\ of\ E_k\ equal\ to\ 2)!...(no.\ of\ E_k\ equal\ to\ n{+}1)!}\ \frac{(n+1)!}{E_1!E_2!...E_{NZE}!}$$

$$= \frac{4}{1}\ \ \frac{5!}{5!} = 4.$$

The monomial q_1q_4 comes from the form $A^1B^4 = A^{E_1}B^{E_2}$, so the coefficient is

$$\frac{4{\cdot}3}{1}\ \ \frac{5!}{1!4!} = 60.$$

The monomial q_2q_3 comes from the form $A^2B^3 = A^{E_1}B^{E_2}$, so the coefficient is

$$\frac{4{\cdot}3}{1}\ \ \frac{5!}{2!3!} = 120.$$

The monomial $q_1^2q_3$ comes from the form $ABC^3 = A^{E_1}B^{E_2}C^{E_3}$, so the coefficient is

$$\frac{4{\cdot}3{\cdot}2}{2}\ \ \frac{5!}{3!} = 240.$$

The monomial $q_1q_2^2$ comes from the form $AB^2C^2 = A^{E_1}B^{E_2}C^{E_3}$, so the coefficient is

$$\frac{4{\cdot}3{\cdot}2}{2}\ \ \frac{5!}{2!2!} = 360.$$

The monomial $q_1^3q_2$ comes from the form $ABCD^2 = A^{e_1}B^{e_2}C^{e_3}D^{e_4}$, so the coefficient is

$$\frac{4{\cdot}3{\cdot}2}{3!}\ \ \frac{5!}{2!} = 240.$$

Check with $n = 2$:

$(A_1 + A_2)^3 = A_1^3 + 3A_2A_1^2 + 3A_2^2A_1 + A_2^3$ reduces to

$q_3 + 3q_1q_2 + 3q_2q_1 + q_3 = 2q_3 + 6q_1q_2$.

The monomial q_3 comes from the forms $A^3 = A^{E_1}$, so the coefficient is

$\frac{3}{1} \ \frac{3!}{3!} = 3$.

The monomial q_1q_2 comes from the forms $AB^2 = A^{E_1}B^{E_2}$, so the coefficient is

$\frac{2 \cdot 1}{1} \ \frac{3!}{2!} = 6$.

Check with $n = 3$:

$(A_1 + A_2 + A_3)^4$ reduces to

$3q_4 + 24q_3q_1 + 18q_2^2 + 36q_2q_1^2$.

The monomial q_4 comes from the form $A^4 = A^{E_1}$, so the coefficient is

$\frac{3}{1} \ \frac{4!}{4!} = 3$.

The monomial q_3q_1 comes from the forms $A^3B^1 = A^{E_1}B^{E^2}$, so the coefficient is

$\frac{3 \ 2 \ 1}{1} \ \frac{4!}{3!} = 24$.

The monomial A_2^2 comes from the forms $A^2B^2 = A^{E_1}B^{E^2}$, so the coefficient is

$\frac{3 \ 2 \ 1}{2} \ \frac{4!}{2!2!} = 18$.

The monomial $A_2A_1^2$ comes from the form $A^2BC = A^{e_1}B^{e^2}C^{e_3}$, so the coefficient is

$\frac{3 \ 2 \ 1}{2} \ \frac{4!}{2!} = 36$.

For $n = 6$:

from the derivative formula for the Schur self-convolution expansion coefficient,

$$b_7 = -\frac{1}{a_0^8}[a_0 a_1^5 a_2 + 10a_0^2 a_1^3 a_2^2 + 5a_0^2 a_1^4 a_3 + 10a_0^3 a_1 a_2^3$$

$$+ 30a_0^3 a_1^2 a_2 a_3 + 10a_0^3 a_1^3 a_4 + 10a_0^4 a_2^2 a_3 + 10a_0^4 a_1 a_3^2 + 20a_0^4 a_1 a_2 a_4$$

$$+ 10a_0^4 a_1^2 a_5 + 5a_0^5 a_3 a_4 + 5a_0^5 a_2 a_5 + 5a_0^5 a_1 a_6 + a_0^6 a_7].$$

With $a_0 = 1$, the change of variables $q_k = k! a_k$ and multiplication by $6 \cdot 7!$ leads to the reduced term

$$(6 \cdot 7!)(30a_1^2 a_2 a_3) = (6 \cdot 7!)\frac{30}{2! \cdot 3!} q_1^2 q_2 q_3$$

from the expansion of $(A_1 + A_2 + ... + A_6)^7$.

The reduced term $q_1^2 q_2 q_3$ is from the form $A^1 B^1 C^2 D^3 = A^{E_1} B^{E_2} C^{E_3} D^4$, so the coefficient, as deduced from the multi-factorial expression, is

$$\frac{6\ 5\ 4\ 3}{2}\ \frac{7!}{2!3!} = 6 \cdot 7! \frac{30}{2!3!},$$

agreeing with the alternative derivative computation above.

With $a_0 = 1$, the change of variables $q_k = k! a_k$ and multiplication by $6 \cdot 7!$ leads to the reduced term

$$(6 \cdot 7!)(10a_1^3 a_2^2) = (6 \cdot 7!)\frac{10}{2!^2} q_1^3 q_2^2$$

from the expansion and reduction of $(A_1 + A_2 + ... + A_6)^7$.

The reduced term $q_1^3 q_2^2$ is from the form $A^1 B^1 C^1 D^2 E^2 = A^{E_1} B^{E_2} C^{E_3} D^{E_4} E^{E_5}$, so the coefficient is

$$\frac{6\ 5\ 4\ 3\ 2}{3!2!}\ \frac{7!}{2!2!} = 6 \cdot 7! \frac{10}{2!^2},$$

which agrees with the alternative derivative calculation.

With $a_0 = 1$, the change of variables $q_k = k! a_k$ and multiplication by $6 \cdot 7!$ leads to the reduced term

$$(6 \cdot 7!)(a_1^5 a_2) = (6 \cdot 7!)\frac{1}{2} q_1^5 q_2$$

from the reduction of the expanded $(A_1 + A_2 + ... + A_6)^7$.

The reduced term $q_1^5 q_2$ is from the form $A^1 B^1 C^1 D^1 E^1 F^2 = A^{E_1} B^{E_2} C^{E_3} D^{E_4} E^{E_5} F^{E_6}$, so the coefficient is

$$\frac{6\ 5\ 4\ 3\ 2}{5!}\ \frac{7!}{2!} = 6 \cdot 7! \frac{1}{2!},$$

which agrees with the alternative derivative calculation.

Multinomial (multi-factorial) coefficients for the monomials of b_n

A direct multinomial formula for the the coefficients of the monomial summands of $-b_n = \frac{K_n^{n-1}}{n-1}$ follows from the formula above for the coefficients of the symbol forms.

For the monomial

$$MBQ_n[e_1, e_2, ..., e_n] = q_1^{e_1} q_2^{e_2} ... q_{n+1}^{e_n}$$

of $-b_n$, we've established that the coefficient of the associated symbol form is

$$\frac{(n-1)_{for\ first\ E_k}(n-2)_{for\ second\ E_k} \cdots}{(no.\ of\ E_k\ equal\ to\ 1)!\ (no.\ of\ E_k\ equal\ to\ 2)! ... (no.\ of\ E_k\ equal\ to\ n)!}\ \frac{n!}{E_1! E_2! ... E_{NZE}!}$$

$$= \frac{(n-1)!}{(n-1-NZE)!} \frac{1}{(no.\ of\ E_k\ equal\ to\ 1)!\ (no.\ of\ E_k\ equal\ to\ 2)! ... (no.\ of\ E_k\ equal\ to\ n)!}\ \ \frac{n!}{E_1! E_2! ... E_{NZE}!}.$$

The first e_1 exponents, i.e., E_1 through E_{e_1}, all have the same value 1; the next e_2 exponents, the value 2; the next e_3, 3; and so on. Consequently,

$$E_1! E_2! ... E_{NZE}! = (1!)^{e_1} (2!)^{e_2} ... (k!)^{e_k} ... (n!)^{e_n}.$$

In addition, the $(no.\ of\ E_k\ equal\ to\ 1)$ is e_1; the $(no.\ of\ E_k\ equal\ to\ 2)$ is e_2; and so on. Consequently,

$$(no.\ of\ E_k\ equal\ to\ 1)!\ (no.\ of\ E_k\ equal\ to\ 2)!...(no.\ of\ E_k\ equal\ to\ n)!$$

$$= e_1!e_2!\cdots e_n!.$$

Making the change of variable $q_k = k!c_k = k!a_k$ and dividing by $(n-1)\ n!$ gives

$$C_n[e_0, e_1, e_2, ..., e_n] = \frac{1}{n-1}\ \frac{(n-1)!}{(n-1-\sum_{k=1}^{n} e_k)!\ e_1!...e_n!} = \frac{(n-2)!}{(n-1-\sum_{k=1}^{n} e_k)!\ e_1!...e_n!}$$

for the coefficient of the monomial

$$M_n[e_0, e_1, e_2, ..., e_n] = a_0^{e_0} a_1^{e_1} a_2^{e_2} ... a_n^{e_n}$$

in

$$-b_n = \frac{K_n^{n-1}}{n-1} = \frac{D_{x=0}^n}{n!}\ \frac{(h(x))^{n-1}}{n-1} = \frac{D_{x=0}^n}{n!}\frac{(a_0+a_1x+a_2x^2+\cdots)^n}{n-1}.$$

(Throughout this set of notes I often have $c_n = a_n$.)

Related analyses in the literature are typically performed with $a_0 = c_0 = q_0 = 1$. If $a_0 \neq 0, 1$, the monomial coefficient can be expressed as

$$C_n[e_0, e_1, e_2, ..., e_n] = \frac{1}{n-1}\frac{(n-1)!}{e_0!e_1!...e_n!} = \frac{(n-2)!}{e_0!e_1!...e_n!}\ \text{with}\ e_0 + e_1 + ... + e_n = n-1.$$

For standard combinatorial interpretations of the multinomial coefficient, see the Wikipedia post "The multinomial theorem".

Examples:

For $n = 7$,

$$-b_7 = \frac{1}{a_0^8}[a_0a_1^5a_2 + 10a_0^2a_1^3a_2^2 + 5a_0^2a_1^4a_3 + 10a_0^3a_1a_2^3 + 30a_0^3a_1^2a_2a_3$$

$$+ 10a_0^3a_1^3a_4 + 10a_0^4a_2^2a_3 + 10a_0^4a_1a_3^2 + 20a_0^4a_1a_2a_4 + 10a_0^4a_1^2a_5$$

$$+ 5a_0^5 a_3 a_4 + 5a_0^5 a_2 a_5 + 5a_0^5 a_1 a_6 + a_0^6 a_7].$$

$a_1^2 a_2 a_3$ gives $e_1 = 2,\ e_2 = 1,\ e_3 = 1$, so $e_1 + e_2 + e_3 = 4$ and

$\frac{(n-2)!}{(n-1-\sum_{k=1}^{n} e_k)!}\ \frac{1}{e_1!\ldots e_n!} = \frac{5!}{2!}\ \frac{1}{2!} = 30$.

$a_1 a_2 a_4$ gives $e_1 = 1,\ e_2 = 1,\ e_4 = 1$, so $e_1 + e_2 + e_4 = 3$ and

$\frac{(n-2)!}{(n-1-\sum_{k=1}^{n} e_k)!}\ \frac{1}{e_1!\ldots e_n!} = \frac{5!}{3!} = 20$.

$a_1 a_3^2$ gives $e_1 = 1,\ e_3 = 2$, so $e_1 + e_3 = 3$ and

$\frac{(n-2)!}{(n-1-\sum_{k=1}^{n} e_k)!}\ \frac{1}{e_1!\ldots e_n!} = \frac{5!}{3!}\frac{1}{2!} = 10$.

$a_1^4 a_2$ gives $e_1 = 4,\ e_2 = 1$, so $e_1 + e_2 = 5$ and

$\frac{(n-2)!}{(n-1-\sum_{k=1}^{n} e_k)!}\ \frac{1}{e_1!\ldots e_n!} = \frac{5!}{1!}\frac{1}{4!} = 5$.

$a_1^5 a_2$ gives $e_1 = 5,\ e_2 = 1$, so $e_1 + e_2 = 6$ and

$\frac{(n-2)!}{(n-1-\sum_{k=1}^{n} e_k)!}\ \frac{1}{e_1!\ldots e_n!} = \frac{5!}{0!}\frac{1}{5!} = 1$.

a_7 gives $e_7 = 1$, so

$\frac{(n-1)!}{(n-\sum_{k=1}^{n} e_k)!}\ \frac{1}{e_1!\ldots e_n!} = \frac{5!}{5!}\frac{1}{1!} = 1$.

For $n = 4$,

$-b_4 = a_1^2 a_2 + a_0 a_2^2 + 2a_0 a_1 a_3 + a_0^2 a_4$.

For $a_1^2 a_2$,

$e_1 = 2,\ e_2 = 1$, and $e_1 + e_2 = 3$, so

$\frac{(n-2)!}{(n-1-\sum_{k=1}^{n} e_k)!}\ \frac{1}{e_1!\ldots e_n!} = \frac{2!}{0!}\frac{1}{2!} = 1$.

For a_2^2,

$e_2 = 2$, so

$$\frac{(n-2)!}{(n-1-\sum_{k=1}^{n} e_k)!} \ \frac{1}{e_1!...e_n!} = \frac{2!}{1!}\frac{1}{2!} = 1$$.

For $a_0 a_1 a_3$,

$e_1 = 1, \ e_3 = 1$, and $e_1 + e_3 = 2$, so

$$\frac{(n-2)!}{(n-1-\sum_{k=1}^{n} e_k)!} \ \frac{1}{e_1!...e_n!} = \frac{2!}{1!}\frac{1}{1!} = 2$$.

Appendix:

Some more identities for the general Schur expansion coefficients related to compositional inverse pairs

Equation. 5.140 on p. 147 of “Enumerative Combinatorics Vol. 2” by Stanley (1999) is

$$[x^n]\frac{(\frac{F^{(-1)}(x)}{x})^k}{k} = [x^n]\frac{(\frac{x}{F(x)})^n}{n},$$

which has a critical, presumably typo, error making the RHS independent of k.

According to Stanley the result "goes back to J. L. Lagrange, Mem. Acad. Roy. Sci. Belles-Lettres Berlin 24 (1770); Oeuvres, Vol. 3 Gauthier-Villars, Paris, 1869, pp. 3-73. It was rediscovered by I. Schur, Amer. J. Math. 69 (1947), 14-26."

And, indeed, the correct version is eqn. 65 on p. 25 of Schur, which is essentially

$$[x^n]\frac{(\frac{F^{(-1)}(x)}{x})^k}{k} = [x^n]\frac{(\frac{x}{F(x)})^{n+k}}{n+k}.$$

With $k = 1$, this becomes

$$[x^n]F^{(-1)}(x) = [x^n]\frac{(\frac{x}{F(x)})^{n+1}}{n+1} = b_n.$$

Equation 5.53 on p. 38 of Stanley is

$$[x^n]\frac{(F^{(-1)}(x))^k}{k} = [x^{n-k}]\frac{(\frac{x}{F(x)})^n}{n} = [x^{-k}]\frac{(\frac{1}{F(x)})^n}{n}.$$

(The first equality is corroborated by spot checks using Wolfram Alpha with $f(x) = e^x - 1$.)

Equivalently, with $h(x) = \frac{x}{f^{(-1)}(x)}$, we have the identity

$f(x) = xh(f(x))$ and the LIF can be expressed as

$$[x^n]\frac{(f(x))^k}{k} = [x^{n-k}]\frac{(h(x))^n}{n}.$$

With $k = 1$, this is the classic Lagrange inversion formula

$$[x^n]f(x) = [x^{n-1}]\frac{(h(x))^n}{n} = [x^{n-1}]\frac{(\frac{x}{f^{(-1)}(x)})^n}{n}.$$

Appendix:

First few partition polynomials of the set $[b][ASP] = [bA]$:

For easy reference, the first few $[b]$ are once again

$b_1 = -c_1$,

$b_2 = -c_2$,

$b_3 = -(c_1c_2 + c_3)$,

$b_4 = -(c_1^2c_2 + 2c_1c_3 + c_2^2 + c_4)$,

$b_5 = -(c_1^3c_2 + 3c_1c_2^2 + 3c_1^2c_3 + 3c_2c_3 + 3c_1c_4 + c_5)$,

and the first few refined associahedra Euler characteristic polynomials are

$A_1(u_1) = (-u_1)$,

$A_2(u_1, u_2) = (2u_1^2 - u_2)$,

$A_3(u_1,u_2,u_3) = (-5u_1^3 + 5u_1u_2 - u_3)$,

$A_4(u_1,u_2,u_3,u_4) = (14u_1^4 - 21u_1^2u_2 + 6u_1u_3 + 3u_2^2 - u_4)$,

so

$bA_1 = -(-u_1) = u_1$

$bA_2 = -(2u_1^2 - u_2) = -2u_1^2 + u_2,$

$bA_3 = -((-u_1)(2u_1^2 - u_2) + (-5u_1^3 + 5u_1u_2 - u_3)) = (2-5)u_1^3 + (-1-5)u_1u_2 + u_3$

$= 7u_1^3 - 6u_1u_2 + u_3,$

$bA_4 = -((-u_1)^2(2u_1^2 - u_2) + 2(-u_1)(-5u_1^3 + 5u_1u_2 - u_3) + (2u_1^2 - u_2)^2$

$+ (14u_1^4 - 21u_1^2u_2 + 6u_1u_3 + 3u_2^2 - u_4))$

$= -30u_1^4 + 36u_2u_1^2 - 8u_3u_1 - 4u_2^2 + u_4$

The last polynomial reduces to $-30t^4 + 36t^3 - 12t^2 + t$, and neither it nor its reverse is in the OEIS.

Appendix::

O.g.f.s for reduced arrays for the special involutive Schur self-convolution expansion polynomials b_n

For the Narayanas,

$\frac{(\frac{1}{1-x}-1+y)^4}{4}$ to order 5 for Narayanas

$= \frac{1}{4}y^4 + y^3x + (y^3 + (3y^2)/2)x^2 + (y^3 + 3y^2 + y)x^3$

$$+ (y^3 + (9y^2)/2 + 3y + 1/4)x^4 + (y^3 + 6y^2 + 6y + 1)x^5 + \cdots$$

$$\frac{1+(t-1)x-\sqrt{1-2(1+t)x+((t-1)x)^2}}{2}$$

$$= \frac{1+(t-1)x-\sqrt{(1+(t-1)x)^2-4tx}}{2}$$

$$= \frac{1-\sqrt{1-\frac{4tx}{(1+(t-1)x)^2}}}{\frac{2}{1+(t-1)x}}$$

$$= tx + tx^2 + (t^2+t)x^3 + (t^3+3t^2+t)x^4 + (t^4+6t^3+6t^2+t)x^5$$

$$+(t^5+10t^4+20t^3+10t^2+t)x^6 + \cdots .$$

Another o.g.f. is

$$\frac{1-\sqrt{1-\frac{4tx}{(1+(t-1)x)^2}}}{\frac{2tx}{1+(t-1)x}}$$

$$= 1 + x + (t+1)x^2 + (t^2+3t+1)x^3 + (t^3+6t^2+6t+1)x^4$$

$$+ (t^4+10t^3+20t^2+10t+1)x^5 + \cdots .$$

An o.g.f. of A091869 with an extra row, 1,

$$\frac{(1+(1-t)x)-\sqrt{(1+(1-t)x)^2-4x(1+(1-t)x)}}{2x}$$

$$= 1 + x + (t+1)x^2 + (t^2+2t+2)x^3 + (t^3+3t^2+6t+4)x^4$$

$$+(t^4+4t^3+12t^2+16t+9)x^5 + \cdots .$$

Another o.g.f. is

$$\frac{1-\sqrt{1-\frac{4tx}{(1+(1-t)x}}}{\frac{2tx}{1+(1-t)x}}$$

$$= 1 + tx + t(t+1)x^2 + t(2t^2+2t+1)x^3 + t(4t^3+6t^2+3t+1)x^4$$

$$+ t(9t^4+16t^3+12t^2+4t+1)x^5 + \cdots .$$

$$\frac{(1+(t-1)x)-\sqrt{((1+(t-1)x))^2-4tx((1+(t-1)x))}}{2}$$

$$= tx + t^2x^2 + (t^3+t^2)x^3 + (2t^4+2t^3+t^2)x^4 + (4t^5+6t^4+3t^3+t^2)x^5$$

$$+ (9t^6+16t^5+12t^4+4t^3+t^2)x^6 + \cdots .$$

Another o.g.f. is

$$\frac{1-\sqrt{1-\frac{4tx}{1+(t-1)x}}}{\frac{2tx}{1+(t-1)x}}$$

$$= 1 + tx + t(t+1)x^2 + t(2t^2+2t+1)x^3 + t(4t^3+6t^2+3t+1)x^4$$

$$+ t(9t^4+16t^3+12t^2+4t+1)x^5 + \cdots .$$

As a Moebius transformation, the Inverse of $y = \frac{tx}{1+(t-1)x}$ is easily found to be $y = \frac{x}{t+(1-t)x}$. This together with a suitable o.g.f. for the Catalan numbers and its inverse allows the inverse of the last o.g.f.

The partial array is

1;

1, 1;

2, 2, 1;

4, 6, 3, 1;

9, 16, 12, 4, 1;

21, 45, 40, 20, 5, 1;

51, 126, 135, 80, 30, 6, 1;

The o.g.f. of A091867 is

$$H(x,t) = \frac{1-\sqrt{1-\frac{4x}{1+(1-t)x}}}{2}$$

$$= x + tx^2 + (t^2+1)x^3 + (t^3+3t+1)x^4 + (t^4+6t^2+4t+3)x^5$$

$$+ (t^5+10t^3+10t^2+15t+6)x^6 + \cdots .$$

The inverse in x about the origin is

$$H^{(-1)}(x,t) = \frac{x}{\frac{1}{1-x}-x+tx} = \frac{x}{1+tx+x^2+x^3+\cdots}$$

$$= \frac{x(1-x)}{1+(t-1)x(1-x)} = \frac{x-x^2}{1+(t-1)(x-x^2)} .$$

For easy reference, the first few rows of the array A091867 are

1,

0, 1,

1, 0, 1,

1, 3, 0, 1,

3, 4, 6, 0, 1,

6, 15, 10, 10, 0, 1,

15, 36, 45, 20, 15, 0, 1,

36, 105, 126, 105, 35, 21, 0, 1

Appendix:

The reciprocal polynomials

Adding a general constant $d_0 \neq 0$, gives

$$\frac{1}{(d_0+d_1x+d_2x^2+d_3x^3+d_4x^4+d_5x^5+...}$$

$$= \frac{1}{d_0}(1) + \frac{1}{d_0^2}(-d_1)x + \frac{1}{d_0^3}(d_1^2 - d_0d_2)x^2 + \frac{1}{d_0^4}(-d_1^3 + 2d_0d_2d_1 - d_0^2d_3)x^3$$

$$+ \frac{1}{d_0^5}(d_1^4 - 3d_0d_2d_1^2 + 2d_0^2d_3d_1 + d_0^2d_2^2 - d_0^3d_4)x^4 + \frac{1}{d_0^6}(-d_1^5 + 4d_0d_2d_1^3$$

$$- 3d_0^2d_3d_1^2 + 2d_0^3d_1d_4 - 3d_0^2d_1d_2^2 + 2d_0^3d_2d_3 - d_0^4d_5)x^5 + ...$$

Summing coefficients of the polynomials in the numerators with like powers of d_0^k generates the row polynomials of the Pascal triangle, the triangle of binomial coefficients A007318, since

$$\frac{1}{d_0+x+x^2+x^3+x^4+x^5+...} = \frac{1}{d_0+\frac{x}{1-x}} = (1-x)\frac{1}{d_0(1-x)+x} = (1-x)\frac{1}{d_0+(1-d_0)x}$$

$$= \frac{1-x}{d_0}\frac{1}{1+\frac{(1-d_0)}{d_0}x} = \frac{1-x}{d_0}\sum_{n\geq 0}(-1)^n(\frac{1}{d_0})^n(1-d_0)^n x^n$$

$$= \frac{1}{d_0} + \frac{1}{d_0}\sum_{n\geq 1}[(-1)^n(\frac{1}{d_0})^n(1-d_0)^n - (-1)^{n-1}(\frac{1}{d_0})^{n-1}(1-d_0)^{n-1}]x^n$$

$$= \frac{1}{d_0} + \frac{1}{d_0}\sum_{n\geq 1}(-1)^{n-1}(\frac{1}{d_0})^{n-1}(1-d_0)^{n-1}[-\frac{1-d_0}{d_0} - 1]x^n$$

$$= \frac{1}{d_0} + \sum_{n\geq 1}(-1)^n(\frac{1}{d_0})^{n+1}(1-d_0)^{n-1}x^n .$$

Appendix:

Introducing a general linear coefficient c_0

Thanks to Schur, we have a self-convolution relating the coefficients $\bar{c}_n$ of the Laurent series

$$\bar{g}(z) = z + \bar{c}_1 + \frac{\bar{c}_2}{z} + \frac{\bar{c}_3}{z^2} + \cdots$$

to the coefficients of the Laurent series

$$\bar{g}^{(-1)}(z) = z + \bar{b}_1 + \frac{\bar{b}_2}{z} + \frac{\bar{b}_3}{z^2} + \cdots .$$

For $n > 0$,

$$-\bar{b}_n = \frac{\bar{K}^n_{n+1}}{n} = \frac{D^{n+1}_{x=0}}{(n+1)!}\frac{(\bar{h}(x))^n}{n} = \frac{D^{n+1}_{x=0}}{(n+1)!}\frac{(1+\bar{c}_1x+\bar{c}_2x^2+\cdots)^n}{n}.$$

For a general non-vanishing c_0, let

$$\bar{g}(z) = \frac{g(z)}{c_0} = z + c_1/c_0 + \frac{c_2/c_0}{z} + \frac{c_3/c_0}{z^2} + \cdots = z + \bar{c}_1 + \frac{\bar{c}_2}{z} + \frac{\bar{c}_3}{z^2} + \cdots$$

and

$$\bar{g}^{(-1)}(z) = z + \bar{b}_1 + \frac{\bar{b}_2}{z} + \frac{\bar{b}_3}{z^2} + \cdots$$

Then

$$g(z) = c_0z + c_1 + \frac{c_2}{z} + \frac{c_3}{z^2} + \cdots$$

and

$$g^{(-1)}(z) = \bar{g}^{(-1)}(z/c_0) = \frac{1}{c_0}\, z + \bar{b}_1 + \frac{c_0\,\bar{b}_2}{z} + \frac{c_0^2\bar{b}_3}{z^2} + \cdots = b_0z + b_1 + \frac{b_2}{z} + \frac{b_3}{z^2} + \cdots,$$

where

$$b_0 = \frac{1}{c_0}$$

and, for $n \geq 1$,

$$b_n(c_0, c_1, ..., c_n) = c_0^{n-1}\bar{b}_n(\bar{c}_1, \bar{c}_2, ..., \bar{c}_n) = c_0^{n-1}\bar{b}_n(\frac{c_1}{c_0}, \frac{c_2}{c_0}, ..., \frac{c_n}{c_0}).$$

Then

$$b_1 = \bar{b}_1(\frac{c_1}{c_0}) = -\frac{c_1}{c_0},$$

and, for $n > 0$,

$$-b_{n+1}(c_0, c_1, ..., c_{n+1}) = c_0^n\, \frac{\bar{K}^n_{n+1}(\bar{c}_1, ..., \bar{b}_{n+1})}{n} = c_0^n\, \bar{b}_{n+1}(\bar{c}_1, \bar{c}_2, ..., \bar{c}_{n+1}),$$

implying

$$-b_{n+1}(c_0,c_1,...,c_{n+1}) = c_0^n \; \frac{D^{n+1}_{x=0}}{(n+1)!} \frac{(1+\bar{c}_1x+....+\bar{c}_{n+1}x^{n+1})^n}{n}$$

$$= \frac{D^{n+1}_{x=0}}{(n+1)!} \frac{(c_0+c_1x+....+c_{n+1}x^{n+1})^n}{n} = \frac{D^{n+1}_{x=0}}{(n+1)!} \frac{(h(x))^n}{n}$$

with

$$h(x) = c_0 + c_1x + ... + c_{n+1}x^{n+1} + \cdots \; .$$

The self-convolutions lead to

$$\frac{(c_0+c_1x+c_2x^2)^1}{1} = c_0 + c_1x + c_2x^2 \cdots$$

and $-b_2 = c_2$;

$$\frac{(c_0+c_1x+c_2x^2+c_3x^3)^2}{2}$$

$$= c_0^2/2 + c_0c_1x + (c_1^2/2 + c_0c_2)x^2 + (c_1c_2 + c_0c_3)x^3 + \cdots$$

and $-b_3 = c_1c_2 + c_0c_3$;

$$\frac{(c_0+c_1x+c_2x^2+c_3x^3+c_4x^4)^3}{3}$$

$$= c_0^3/3 + c_0^2c_1x + (c_0c_1^2 + c_0^2c_2)x^2 + (c_1^3/3 + 2c_0c_2c_1 + c_0^2c_3)x^3$$

$$+ (c_2c_1^2 + 2c_0c_3c_1 + c_0c_2^2 + c_0^2c_4)x^4 + \cdots$$

and $-b_4 = c_2c_1^2 + 2c_0c_3c_1 + c_0c_2^2 + c_0^2c_4$; and

$$\frac{(c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5)^4}{4}$$

$$= c_0^4/4 + ... + (c_0c_1^3 + 3c_0^2c_2c_1 + c_0^3c_3)x^3 + (c_1^4/4 + 3c_0c_2c_1^2 + 3c_0^2c_3c_1 + 3/2c_0^2c_2^2 + c_0^3c_4)x^4$$

$$+ (c_2c_1^3 + 3c_0c_3c_1^2 + 3c_0c_2^2c_1 + 3c_0^2c_4c_1 + 3c_0^2c_2c_3 + c_0^3c_5)x^5 + \cdots$$

and

$$-b_5 = c_2 c_1^3 + 3c_0 c_3 c_1^2 + 3c_0 c_2^2 c_1 + 3c_0^2 c_4 c_1 + 3c_0^2 c_2 c_3 + c_0^3 c_5.$$