In the Flow

Infinigins, fixed point equations, and flows in functional iteration and renormalization

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The Abel-Graves-Lie theory of flow equations and iterated infinitesimal generators, or infingens, (IIGs) form a common thread weaving through functional iteration (FI) in complex dynamics (CD) and the vector fields and integral curves of classical and quantum mechanics, in particular the renormalization group flow (RGF) in statistical and quantum physics (see Zinn-Justin, Wilson, Barenblatt, Zeidler, and Weinberg). Below are some observations. First the analysis, and then some history related to certain fairly well-known personages (cameos by Archimedes) and their less well-known contributions to the subject.

Two fixed-point equations central to FI, or compositional iteration, in CD--Abel's and Schröder's (see Alexander)--are counterparts to the flow equation for the beta function $\beta(y)$ of the RGF theory of quantum field theory (QFT)

$$\mu \frac{dg}{d\mu} = \frac{dh}{dx} = \beta(h(x)).$$

This defines a flow field, or vector field for integral curves, so prominent in pure and applied mathematics. This is eqn. (21) on pg. 7, (52) and (56) on pg. 14, and (95) on pg. 20 of "Wilsonian renormalization, differential equations and Hopf algebras" by Krajewski and Martinetti (K & M) with the non-autonomous version eqn. (46) on pg. 12 and (108) on pg. 22. See also comments on pages 111 and 170-1 of "Renormalization From Lorentz to Landau (and Beyond)" compiled by Brown.

With $h^{(-1)}(y)$ the (analytic or formal series) compositional inverse of $h(x)$ on a neighborhood about the point $(x_c, y_c) = (h^{(-1)}(y_c), h(x_c))$ and with

$$\beta(y) = \frac{1}{D_y \ h^{(-1)}(y)} = \frac{1}{(h^{(-1)})'(y)},$$

then

$$\partial_x = \frac{\partial}{\partial x} = \frac{\partial}{\partial h^{(-1)}(y)} = \beta(y)\ \frac{\partial}{\partial y} = \beta(y)\ \partial_y.$$

Define a flow function $F(t,y)$ formally via the IIGs as

$$\exp[\ t\ \beta(y)\ \delta_y\ ]\ y = \exp[\ t\ \delta_x]\ h(x) = h(x+t) = h(h^{(-1)}(y)+t) = F(t,y).$$

The equality is to be understood as local for the neighborhood of analyticity and bijectivity about the point $(x_c, y_c) = (h^{(-1)}(y_c), h(x_c))$ or more generally as an equivalence under analytic continuation. Here I've glossed over the issue of non-bijective functions and the multiplicity of their inverse functions and, therefore, the flow function, but this is illustrated for a couple of examples in Appendices 1 and 7, my response to the MO-Q "Do the complex iterates of functions have any meaning?", and some other posts.

A flow function appears in Schröder's 1870 paper "Über iterirte Functionen" (pg. 303). The connections among the flow function $F(x,y) = h(x + h^{(-1)}(y))$ where x and $h^{(-1)}(y)$ are regarded as independent here; Abel's, Böttcher's, and Schröder's equations; and extremals of functionals are concisely presented in "Variational aspects of the Abel and Schröder functional equations" by McKiernan. Relationships among four different differential equations, the flow function, and a translational functional equation are given in "Some differential equations related to iteration theory" by Aczel and Granau. Further similar analysis as well as a historical perspective is presented in "A survey on the hypertranscendence of the solutions of the Schröder's, Böttcher's and Abel's equations" by Fernandes. For additional info on these functional / fixed point equations in CD, see the appendices on Schröder's and Frege's forays into FI below and the Wikipedia articles on Julia sets, fractals, and the Mandelbrot set. For details of the relations among flow equations, CD, and QFT, see the paper by Curtwright and Zachos cited below and the following discussion, the appendix on Ovsyannikov's diff eq, and other appendices below. OEIS A145271, A133437, links therein, and the section Related Stuff below contain other refs.

Continuing the analysis, differentiation gives

$$\partial_{x=0}\ F(x,y) = h'(h^{(-1)}(y)) = \beta(y) = \frac{1}{(h^{(-1)})'(y)},$$

in agreement with the inverse function theorem, and substitution gives

$$h'(x) = \beta(h(x)) = \frac{1}{(h^{(-1)})'(h(x))}.$$

The flow function has the translation group property, at least in the immediate neighborhood of analyticity about $(x_c, y_c) = (h^{(-1)}(y_c), h(x_c))$,

$$F(x, F(u, y)) = F(x + u, y),$$

and Abel's equation in the Wikipedia article

$$\alpha(f(x)) = \alpha(x) + 1$$

is a special case of this translation equation. Changes of variables transform Abel's equation into Schröder's, Böttcher's, Julia's, and others encountered in CD. (Compare the group property here with the group transformation / functional equation (A.11) for renormalization discussed by Shirkov on pg 181 of Brown.)

Note that

$$\frac{\partial F(x,y)}{\partial x} - \beta(y)\frac{\partial F(x,y)}{\partial y}$$

$$= \frac{\partial}{\partial x}\ \exp[\ x\ \beta(y)\ \partial_y\ ]\ y - \beta(y)\frac{\partial F(x,y)}{\partial y}$$

$$= \beta(y)\frac{\partial F(x,y)}{\partial y} - \beta(y)\frac{\partial F(x,y)}{\partial y} = 0.$$

This also follows, without resorting to the iterated IG, from

$$\partial_y\ F(x,y) = h'(x + h^{(-1)}(y))\ (h^{(-1)})'(y) = \frac{1}{\beta(y)}\partial_x\ F(x,y).$$

Consequently, $(1, -\beta(y))$ are the components of a vector orthogonal to the gradient of F and, therefore, tangent to the contour of F at (x, y), and we have all the trappings of geometric Lie theory and geometric optics with the tangency property

$$\frac{\partial F(x,y)}{\partial x} - \beta(y)\frac{\partial F(x,y)}{\partial y} = 0.$$

Reviewing the derivation of this tangency equation through diffeomorphism, or ‘coordinate’ transformation, it should be no surprise that solutions to this diff eq reflect that for the diff eq for simple translation symmetry,

$$\frac{\partial W(x+z)}{\partial x} - \frac{\partial W(x+z)}{\partial z} = 0$$

with $z = h^{(-1)}(y)$, and that since the RGF method addresses invariance under certain physical symmetries at critical points, this tangency equation pops up in this scenario. (See also the sections '0.5 Self-similarities and traveling waves' and '7.1 Solutions of traveling wave type' in the book Scaling, Self-Similarity, and Intermediate Asymptotics by Barenblatt.)

Also note

$$F(0, y) = y$$

and

$$\int \frac{dy}{\beta(y)} = \int (h^{-1})'(y)\, dy = h^{(-1)}(y) + \ constant.$$

We see the iterated IGs $(\beta(y) D_y)^n$ occurring in both QFT and CD in relation to flow equations, so no surprise that the methodology in the two fields overlap. (Compare the integral with that of (A.2) discussed by Shirkov on pg. 171 of Brown.)

In the appendices, I reprise the various flow equations from several perspectives.

(For an encoding in differential forms of several of the relations above, see sections 4.17 and 4.25 of "Geometrical Methods of Mathematical Physics" by Schutz.)

Curtwright and Zachos, in "Renormalization Group Functional Equations," give an excellent overview of the relationship between Schröder's functional equation and the RG beta function with examples worked out. In Appendix VIII on the Newtonian trajectory as a transport of data, they present a connection to Newtonian dynamics and the tangency equation just above. (There is another link, not presented in their paper, to physical dynamics via the relation of the inviscid Burgers-Hopf differential equation with compositional inversion.)

In their appendix, C & Z delineate the connections between a simple physical model and the flow equations. With E and $U(x)$ being the total energy and potential energy of a particle in motion in a conservative potential field (i.e., the total energy is conserved), the kinetic energy is

$$T = E - U = p^2/2m = mv^2/2$$

where the momentum is p; the mass, m; and the velocity, v. Then the amount of time Δt taken to traverse a trajectory from a fixed initial position x_i at a fixed time t_i to a fixed final position at x_f at time t_f with the final time determined by the velocity of the particle along its path to the final fixed position is

$$\int_{x_i}^{x_f} \frac{dx}{\sqrt{\frac{m}{2}\ (E - U(x))^2}} = \int_{x_i}^{x_f} \frac{dx}{v(x)}$$

$$= \int_{x_i(t)}^{x_f(t)} \frac{dx}{\dot{x}(t)} = t_f - t_i = \Delta(t).$$

This should be interpreted as dx and $v(x)$ having the same sign, as dx and $v(x)$ being parallel vectors, or, more precisely, as dx being the line element at that part of a trajectory for which $v(x)$ is the tangent vector. For more discussion on C & Z's presentation, see Appendix 6.

More precisely, assign $x = h(t)$ and $t = h^{(-1)}(x)$. Then the inverse function theorem

$(h^{(-1)})'(x) = \frac{1}{h'(h^{(-1)}(x))}$

implies the string of equalities

$dt = dh^{(-1)}(x) = (h^{(-1)})'(x)\ dx = \frac{dt}{dx}\ dx = \frac{1}{\frac{dx}{dt}}\ dx$

$= \frac{1}{h'(t)}\ dx = \frac{1}{h'(h^{(-1)}(x))}\ dx$

$= \frac{1}{\dot{x}(t)}\ dx = \frac{1}{v(x)}\ dx,$

in which I identify

$v(x) = h'(h^{(-1)}(x)).$

Notwithstanding H. G. Wells' spin and quantum machinations, $t = h^{(-1)}(x)$ must always be increasing along the trajectory, but, of course, the particle can reverse directions of travel, so bijectivity between position and time is maintained typically only locally along the trajectory. In other words, $x = h(t)$ is classically indeed a global function, but $t = h^{(-1)}(x)$ is not a global function necessarily but typically only over restricted intervals of time between turning points. In other words there will typically be a patchwork of bijective functions characterizing $h(t)$ and its local inverse. See some examples below and in the pdfs in my posts "Local compositional inversion of f(x) = x / (1+ax+bx^2)" and "A Taste of Moonshine in Free Moments".

The physical solution is governed by the equivalent mathematical formalisms of Newtonian, Hamiltonian, Lagrangian, and Hamilton-Jacobi dynamics, allied to the calculus of variations, and the physical ideas surrounding Maupertuis' principle of least 'action', Fermat's principle of least time (more accurately, stationary principles), Huygen's principle, and geometrical optics (see, e.g., Feynman's Lectures on Physics).

Reprising some of the formulas for the infinigen above with a change in variables to avoid confusion:

$$\beta(y) = \frac{1}{(h^{(-1)})'(y)} = h'(h^{(-1)}(y)),$$

$$\beta(h(\omega)) = h'(\omega) = \frac{1}{(h^{(-1)})'(h(\omega))},$$

$$\exp[\ \omega\ \beta(y)\ \delta_y\ ]\ y = \exp[\ \omega\ \delta_x]\ h(x) = h(\omega + x) = h(\omega + h^{(-1)}(y)) = F(\omega, y),$$

$$F(\omega, F(\gamma, y)) = F(\omega + \gamma, y),$$

$$\frac{\partial F(\omega, y)}{\partial \omega} - \beta(y) \frac{\partial F(\omega, y)}{\partial y} = 0,$$

$$F(0, y) = y,$$

and

$$\int \frac{dy}{\beta(y)} = \int\ (h^{-1})'(y)\ dy = h^{(-1)}(y).$$

Of course, we are free to switch the roles of the forward and inverse functions to obtain a mirror panoply of dual equations:

$$\exp[\gamma\ \bar{g}(z)\ \partial_z)\ z = h(\gamma + h^{(-1)}(z)) = \bar{F}(\gamma, z)$$

and

$$\frac{\partial \bar{F}}{\partial \gamma} - \bar{g}(z)\ \frac{\partial \bar{F}}{\partial z} = 0$$

with

$\bar{g}(z) = \frac{1}{(h^{(-1)})'(z)},$

and since

$h(h^{(-1)}(u)) = u = h^{(-1)}(h(u)),$

derivation gives the inverse function theorem

$h'(h^{(-1)}(u)) \cdot (h^{(-1)})'(u) = 1 = (h^{(-1)}(h(u)) \cdot h'(u),$

which implies that the graphs of the pair of inverse functions $y = h(x)$ and $y = h^{(-1)}(x)$ are reflections of each other through the bisecting diagonal line $y = x$ (e.g., graph $y = x^2$ and $y = \pm\sqrt{x}$.)

The two vectors with components $(1, g(x)$ and $(-1, \bar{g}(x))$ are then orthogonal to each other, implying the flow lines of the two fields are mutually orthogonal as well, a feature central to the theory of geometric optics and its applications to quantum physics.

To illustrate these interconnections, identify $\beta \to v$ and $y \to x$. Then $x = h(\omega)$ while $\omega = h^{(-1)}(x)$ and

$$v(x) = \frac{1}{(h^{(-1)})'(x)} = h'(h^{(-1)}(x)),$$

$$v(h(\omega)) = h'(\omega) = \frac{1}{(h^{(-1)})'(h(\omega))},$$

$$\exp[\ u\ v(x)\ \delta_x\ ]\ x = \exp[\ u\ \delta_\omega\ ]\ h(\omega) = h(u+\omega) = h(u+h^{(-1)}(x)) = F(u,x),$$

$$F(\omega, F(\gamma, x)) = F(\omega+\gamma, x),$$

$$\frac{\partial F(\omega,x)}{\partial \omega} - v(x)\frac{\partial F(\omega,x)}{\partial x} = 0,$$

$$F(0,x) = x,$$

and

$$\int \frac{dx}{v(x)} = \int (h^{-1})'(x)\ dx = h^{(-1)}(x).$$

Identifying ω with the time t, then $x = h(t)$ and $h^{(-1)}(x) = t$, with plenty of choices for the function h over the duration Δt. Along any path for general $x = h(t))$, tangency is maintained, i.e., noting that $\partial_\alpha W(\alpha+\beta) = \partial_\beta W(\alpha+\beta)$ for any function,

$$\begin{aligned} 0 &= \frac{\partial F(\omega,x)}{\partial \omega} - v(x)\frac{\partial F(\omega,x)}{\partial x} \\ &= \frac{\partial h(\omega+t)}{\partial \omega} - v(x)\frac{\partial h(\omega+t)}{\partial x} \\ &= \frac{\partial h(\omega+t)}{\partial t} - v(x)\frac{\partial h(\omega+t)}{\partial x}, \end{aligned}$$

so

$$\frac{\partial h(\omega+t)}{\partial t} - v(x)\frac{\partial h(\omega+t)}{\partial x} = 0,$$

which is equation (85) of C & Z, their one-dimensional ‘Gell-Mann–Low transport equation’ for $x \to x_0$ and $h(t) \to x(t, x_0)$.

With $x(t) = h(t)$, the tangency condition becomes

$$\dot{x}(t) = v(x(t)) = \pm\sqrt{\frac{2}{m}\ (E - U(x))},$$

so the flow field and potential energy track each other and for a conservative potential

$$force = mass \cdot acceleration = m \cdot \ddot{x}(t)$$

$$= m \cdot \frac{\partial v(x(t))}{\partial t} = m \; \frac{\partial v}{\partial x} \; \dot{x}(t)$$

$$= m \; [\frac{1}{2}\frac{2}{m}\frac{1}{\pm\sqrt{\frac{2}{m} \; (E-U(x))}} \; (-\frac{\partial U(x)}{\partial x})] \; [\pm\sqrt{\frac{2}{m} \; (E-U(x))}]$$

$$= -\frac{\partial U(x)}{\partial x}.$$

giving the fundamental equation of Newtonian classical mechanics

$$force = m \; \ddot{x}(t) = -\frac{\partial U(x)}{\partial x}.$$

Then with the Hamiltonian

$$H = T + U = \frac{p^2}{2m} + U(q),$$

we can identify

$$\dot{q} = \frac{\partial q}{\partial t} = \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} = v(x(t))$$

and

$$\dot{p} = \frac{\partial p}{\partial t} = -\frac{\partial H}{\partial x} = -\frac{\partial U}{\partial x} = F = m \; \ddot{x} = m\frac{\partial v(x(t))}{\partial t}.$$

The dual Lagrangian formulation is

$$L(x, \dot{x}) = T(\dot{x}) - U(x),$$

so

$$\frac{\partial L}{\partial \dot{x}} = \frac{\partial T}{\partial \dot{x}} = m \; \dot{x} = p = m \; v(x(t)),$$

and

$$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = F = m\,\ddot{x} = \frac{dp}{dt} = \frac{d}{dt}\;m\;v(x(t)),$$

and the Euler-Lagrange equation is satisfied

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \frac{d}{dt}\;m\;v(x(t)) - \frac{d}{dt}\;m\;v(x(t)) = 0$$

and is equivalent to

$$F = \frac{d}{dt}\;p = -\frac{\partial U}{\partial x}.$$

The Hamilton-Jacobi formalism (HJF) of classical geometric optics and mechanics provides another approach to the physics of motion (see also Chapter 3 of these course notes by Helliwell and Sahakian).

A controlling differential equation of the HJF (see p. 417 of "The Variational Principles of Mechanics" by Lanczos or Cline?) is

$$\frac{\partial S}{\partial t} + W(x, \frac{\partial S}{\partial x}, t) = 0,$$

where W is Hamilton's characteristic / principle function, S is the Maupertuis abbreviated action, and the conjugate momentum is

$$\frac{\partial S}{\partial x} = p.$$

With $W(x,p,t) = H(x,p,t)$, the conjugate momentum for time is the negated energy, i.e.,

$$\frac{\partial S}{\partial t} = -E,$$

and

$$\frac{\partial S}{\partial x} = \frac{\partial W}{\partial x} = p = m\;\dot{x}(t) = m\;v(x(t)),$$

so

$$\frac{\partial S}{\partial t} + W(x, \frac{\partial S}{\partial x}, t) = -E + W(x, \frac{\partial S}{\partial x}, t)$$

$$= -E + H(x,p) = -E + \frac{1}{2m}\, p^2 + U(x) = -E + E = 0,$$

consistent with

$$\frac{\partial^2 S}{\partial t\, \partial x} = \dot{p} = -\frac{\partial U(x)}{\partial x}.$$

For a particle moving in a conservative potential, i.e., with $W(x,p) = H(x,p)$ independent of t and, therefore, with constant energy E, the Hamiltonian characteristic function is

$$E = W(x, \frac{\partial S}{\partial x}) = T + U = \frac{1}{2m}(\frac{\partial S}{\partial x})^2 + U(x),$$

so

$$S = \int \sqrt{2m\,(E - U(x))}\, dx - Et.$$

Then the variation of S with E

$$\frac{\partial S}{\partial E} = \int \frac{1}{\sqrt{\frac{2}{m}\,(E - U(x))}}\, dx - t$$

$$= \int \frac{1}{v(x)}\, dx - t,$$

vanishes for $v(x(t)) = \dot{x}(t)$, i.e., the velocity vector remains tangent to the path the particle follows and this tangent is determined by the conservative potential. For example, for a free particle moving without the influence of any forces, characterized by $U(x) = 0$, let the initial and final positions, x_i and x_f, of the particle be fixed at fixed times t_i and t_f. Then the average velocity of the particle traveling, whether along a geodesic or not, between those two fixed space-time events is $\bar{\dot{x}} = \frac{x_f - f_i}{x_f - t_i} = \frac{\Delta x}{\Delta t} = \bar{V}$. The deviation at time t in kinetic energy of a particle moving along any curve from that of the same particle moving with the constant velocity $\bar{V}$ along a straight line, i.e., along a geodesic, is, with $\dot{x}(t) = V(t) = v(x(t))$,

$$\delta T(t) = \tfrac{m}{2}\ (\ (V(t))^2 - \bar{V}^2\ )$$

and the deviation averaged over the fixed duration Δt is

$$\overline{\delta T} = \frac{m}{2}\ (\ \overline{(V(t))^2} - \bar{V}^2\ ) = \frac{m}{2\Delta t}\int_{t_i}^{t_f}\ (\ (V(t))^2 - \bar{V}^2\ )\ dt$$

but also

$$\frac{m}{2\Delta t}\int_{t_i}^{t_f}\ (\ V(t) - \bar{V}\ )^2\ dt$$

$$= \frac{m}{2\Delta t}\int_{t_i}^{t_f}\ (\ (V(t))^2 - 2V(t)\bar{V} + \bar{V}^2\ )\ dt$$

$$= \frac{m}{2\Delta t}\int_{t_i}^{t_f}\ (\ (V(t))^2 - \bar{V}^2\ )\ dt = \frac{m}{2}(\overline{V^2} - \bar{V}^2) = \overline{\delta T},$$

so

$$\overline{\delta T} = \frac{m}{2\Delta t}\int_{t_i}^{t_f}\ (\ (V(t))^2 - \bar{V}^2\ )\ dt = \frac{m}{2}(\overline{V^2} - \bar{V}^2)$$

$$= \frac{m}{2\Delta t}\int_{t_i}^{t_f}\ (\ V(t) - \bar{V}\ )^2\ dt \geq 0,$$

implying any variation of the path from a straight line, which would vary the tangent and therefore the velocity of a particle from the average, or any acceleration / deceleration along the straight path, would lead to an averaged kinetic energy larger than that of the a particle moving along a straight line with a constant velocity that satisfies the initial and final space-time conditions, so only for a particle moving in a straight line with constant velocity is the minimum average kinetic energy achieved. Consequently, this physical solution is achieved when the action is a minimum, which implies $\frac{\partial S}{\partial E}$ vanishes for the free particle; that is,

$$\frac{\partial S}{\partial E} = \int_{x_i=x(t_i)}^{x_f=x(t_f)} \frac{1}{\sqrt{\frac{2}{m}\,(E-U(x))}}\,dx - \Delta t$$

$$= \frac{\partial S}{\partial T} = \int_{x_i=x(t_i)}^{x_f=x(t_f)} \frac{1}{\sqrt{\frac{2}{m}\,T(x)}}\,dx - \Delta t$$

$$= \int_{x_i=x(t_i)}^{x_f=x(t_f)} \frac{1}{v(x)}\,dx - \Delta t = \frac{1}{\bar{v}}\Delta x - \Delta t = 0.$$

This is an example of one form of the principle of stationary action for the action defined for a potential independent of t as

$$S = \int_{x_i(t_i)}^{x(t)} p(x)\,dx - E\,(t-t_i) = \int_{x_i(t_i)}^{x(t)} \sqrt{2m\,T(x)}\,dx - E\,(t-t_i)$$

$$= \int_{x_i(t_i)}^{x(t)} \sqrt{2m\,(E-U(x)}\,dx - E\,(t-t_i)$$

and

$$\frac{\partial S}{\partial E} = \int_{x_i(t_i)}^{x(t)} \frac{1}{\sqrt{\frac{2}{m}\,(E-U(x)}}\,dx - (t-t_i)$$

$$= \int_{x_i(t_i)}^{x(t)} \frac{1}{\sqrt{\frac{2}{m}\,T(x)}}\,dx - (t-t_i)$$

$$= \int_{x_i(t_i)}^{x(t)} \frac{m}{p(x)}\,dx - (t-t_i)$$

$$= \int_{x_i(t_i)}^{x(t)} \frac{1}{v(x)}\,dx - (t-t_i),$$

so

$\frac{\partial S}{\partial E} = 0$ implies

$$\int_{x_i(t_i)}^{x(t)} \frac{1}{v(x)}\,dx = (t-t_i),$$

where

$$v(x) = \dot{x}(t) = \sqrt{2m\ (E - U(x))}.$$

This is all consistent with the local inverse pair $x = h(t)$ and $t = h^{(-1)}(x)$ when local bijectivity holds and the associated derivative relation

$$\beta(y) = v(x(t)) = \dot{x}(t) = h'(t)$$

$$= \frac{1}{((h)^{(-1)})'(h(t))} = \frac{1}{((h)^{(-1)})'(x)},$$

so

$$\int_{x_i(t_i)}^{x(t)} \frac{1}{v(x)}\ dx = \int_{t_i}^{t} \frac{1}{h'(t)}\ h'(t)\ dt = \int_{t_i}^{t} dt = t - t_i,$$

or

$$\int_{x_i(t_i)}^{x(t)} \frac{1}{v(x)}\ dx = \int_{x_i(t_i)}^{x(t)} ((h)^{(-1)})'(x)\ dx = h^{(-1)}(x)\ |_{x_i}^{x(t)} = t - t_i.$$

The physics enters through

$$v(x(t)) = \sqrt{2m\ T(x)} = \sqrt{2m\ (E - U(x))}.$$

A test particle then can be used to map the flow lines and therefore the conservative potential.

Note , for the initial condition $x(0) = 0$,

$$\exp[t\ v(x)\ \partial_x\ ]x\ |_{x=0} = h(t + h^{(-1)}(x))\ |_{x=0} = h(t) = x(t)$$

$$= \exp[t\ \sqrt{2m\ (E - U(x))}\ \partial_x]\ x\ |_{x=0},$$

and for our free particle with constant total energy $E = \frac{v_0^2}{2m}$ all kinetic,

$$x(t) = \exp[t\ v_0\ \partial_x]\ x\ |_{x=0} = (x + vt)\ |_{x=0} = v_0\ t.$$

If we evaluate the infinigen series at x_0, we have $x(t) = v_o t + x_o$.

To illustrate the equivalence of the Newtonian, Hamiltonian, Lagrangian, and Hamilton-Jacobi formalisms, consider the iconic harmonic oscillator with the potential $U = \frac{1}{2}k\ x^2$ and spring constant k. Newton's equation becomes

$$F = m\ \ddot{x} = -\frac{\partial U}{\partial x} = -k\ x.$$

With initial conditions $\dot{x}(0) = v(0) = v_0 = 0$ at $x = x(0) = x_0 > 0$, the solution is

$$x(t) = x_0\ \cos(\sqrt{\frac{k}{m}}\ t).$$

The Hamiltonian formalism gives

$$H(x,p) = \frac{p^2}{2m} + \frac{1}{2}k\ x^2,$$

$$-\frac{\partial H}{\partial x} = -kx = F = \dot{p} = -k\ x_0\ \cos(\sqrt{\frac{k}{m}}\ t),$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} = -\sqrt{\frac{k}{m}}\ x_0\ \sin(\sqrt{\frac{k}{m}}\ t) = -v_M\ \sin(\sqrt{\frac{k}{m}}\ t),$$

where $v_M = max\ speed = x_0\ \sqrt{\frac{k}{m}}$ and $E = T + U = \frac{1}{2}k\ x_0^2 = \frac{m\ v_M^2}{2}$.

The Lagrangian formalism gives

$$L(x,\dot{x}) = \frac{m\ \dot{x}^2}{2} - \frac{1}{2}k\ x^2,$$

$$\frac{\partial L}{\partial x} = -kx = F = \dot{p} = -k\ x_0\ \cos(\sqrt{\frac{k}{m}}\ t),$$

$$\frac{\partial L}{\partial \dot{x}} = m\ \dot{x} = p = -\sqrt{k\ m}\ x_0\ \sin(\sqrt{\frac{k}{m}}\ t),$$

and the Euler-Lagrange equation is

$$\frac{\partial L}{\partial x} - \frac{d}{dt}\frac{\partial L}{\partial \dot{x}} = \dot{p} - \dot{p} = 0.$$

The Hamilton-Jacobi formalism gives

$$S = \int \sqrt{2m\,(E - U(x))}\,dx - E\,t$$

$$= \int \sqrt{2m\,(\frac{1}{2}k\,x_0^2 - \frac{1}{2}k\,x^2)}\,dx - \frac{1}{2}k\,x_0^2\,t$$

$$= \sqrt{m\,k}\,x_0 \int \sqrt{1 - (\frac{x}{x_0})^2}\,dx - \frac{1}{2}k\,x_0^2\,t$$

$$= \frac{1}{2}\,\sqrt{m\,k}\,x_0\,[\,x\sqrt{1 - (\frac{x}{x_0})^2} + x_0\,\arcsin(\frac{x}{x_0}) + C\,] - \frac{1}{2}k\,x_0^2\,t$$

$$= \frac{m}{2}\,\sqrt{\frac{k}{m}}\,x_0^2\,[\,\cos(\sqrt{\frac{k}{m}}\,t)\,\sin(\sqrt{\frac{k}{m}}\,t) + \arcsin(\cos(\sqrt{\frac{k}{m}}\,t)) + C\,] - \frac{1}{2}k\,x_0^2\,t$$

$$= \frac{m}{2}\,\omega\,x_0^2\,[\,\cos(\omega\,t)\,\sin(\omega\,t) + \arcsin(\cos(\omega\,t)) + C - \omega\,t\,].$$

This is discussed in terms of RG flow in “RG flows, cycles, and c-theorem folklore” by Curtright, Jin, and Zachos.

For the harmonic oscillator with our initial conditions, the variation stationary condition

$$\frac{\partial S}{\partial E} = \frac{1}{\sqrt{\frac{k}{m}}\,x_0}\int \frac{1}{\sqrt{1 - (\frac{x}{x_0})^2)}}\,dx - t = 0$$

when integrated gives

$$\frac{1}{\sqrt{\frac{k}{m}}}\;\arcsin(\frac{x}{x_0}) + C - t = 0,$$

and inverting.

$$x = x_0 \ \sin[\frac{k}{m}(t - C)]$$

where the constant C is determined by $x(0) = 0$, so

$$x(t) = x_0 \ \cos(\frac{k}{m} \ t) = x_0 \ \cos(\omega \ t).$$

For more on the action and the associated flow function, see Appendix 4.

From the intro to "Renormalization Group Functional Equations" by Curtright and Zachos:

The renormalization group (RG) of Gell-Mann and Low [14], and of Stueckelberg and Petermann [24], has an elegant mathematical expression in terms of the functional conjugation (FC) methods of Ernst Schröder [23]. This expression provides a powerful tool to describe the behavior of physical systems under either infinitesimal or finite, perhaps large, changes in scale. While this fact is often overlooked, and not usually invoked in the solution of various problems posed in the RG framework, it is readily apparent upon reading [14] (see especially Appendix B; also see [19]) and surveying the literature on functional equations [16]. Moreover, it may be profitable to bear in mind the logical connections between these two subjects when considering the step-scaling approach in lattice gauge theory [4, 20], where the power and utility of the methods are manifest.

K & M present the following comparison between structures/concepts in the theory of flow equations encountered in CD and QFT.

- rooted trees ⇔ Feynman diagrams
- nonlinear analysis ⇔ perturbative path integrals
- fixed point equations ⇔ renormalization group equation
- powers of nonlinear operators X^t ⇔ background field technique $A^\gamma(S)$
- $x' = \sum_t \frac{X^t(x)}{S_t}$ ⇔ $S'(\phi) = \log(path\ integral) = \sum_\gamma \frac{A^\gamma(S)}{S_\gamma}[\phi]$
- composition ⇔ successive integrations

There are several partition polynomials associated with iterated IGs and flow equations. They are accompanied by diverse combinatorial models, such as lattice paths, trees, polytopes, and polygons, and, sInce compositional inversion plays an integral role in characterizing binomial Sheffer polynomial sequences, these polynomials weasel their way into a multitude of niches in operator theory and higher algebra, classical and quantum.

The iterated IGs are characterized in normally-ordered form by the Comtet-Scherk partition polynomials of A139605. A refinement of these are the Connes-Moscovici (CM) partition polynomials for the derivatives $D_y^n\ h(y)$ in terms of the derivatives $D_h^n\ \beta(h(y))$, called elementary differentials (EDs), when β and h are vector functions. The coefficients, the CM weights, of these polynomials are presented in A139002. K & M elaborate on low order examples on p. 14.

Add the assumptions of analyticity in a neighborhood about the origin $(x,y)=(0,0)$ and $h^{(-1)}(0)=0$, then $h(x)=F(x,0)$, implying the IIGs acting on y and evaluated at $y=0$ give the series coefficients of different reps of the Lagrange inversion formula (LIF) for compositional inversion, series reps which depend on those for the source, β, and the target series. The actions $(\beta(y)\ D_y)^n\ y = (\beta(y)\ D_y)^{n-1}\ \beta(y)$ are characterized by the refined Eulerian partition polynomials of A145271, with the changes in notation $x \to t$, $y \to x$, and $\beta(y) \to g(x)$, which then give the e.g.f. in t for $F(t,x)=h(t+h^{(-1)}(x))$. (See my answer to the MO-Q “Formula for n-th iteration of dx/dt=B(x)” for more details.)

Schröder, in his 1870 paper "Ueber unendlich viele Algorithmen zur Auflösung der Gleichungen" (English translation “On infinitely many algorithms for solving equations” by Stewart, with some errors in transcription of some polynomials) on Newton's fixed point method for finding the zeros of functions and FI, uses the action $(\frac{1}{f'(z)}\ D_z)^n\ z$ to generate a flow function containing the partition polynomials of the LIF series of A134685 (evaluated at general z rather than $z=0$, more on this in Appendix 2). These partition polynomials give the antipode of the Faa di Bruno combinatorial Hopf algebra--not so surprising as Brouder shows in "Trees, renormalization, and differential equations" that if you understand the calculus of differentiation you understand fundamental aspects of Hopf algebras.

Heinrich Scherk introduced the normal-ordered form of iterated IGs in his dissertation in 1823 (see A139605), one of the two mathematically talented Graves brothers, Charles, (both colleagues of Cayley, Sylvester, and Hamilton) published the generalized shift identity for the flow function above in "On a generalization of the symbolic statement of Taylor's theorem" in 1852, and then Cayley showed in 1857 that the EDs of the CM partition polynomials are in bijection with the trees of 'naturally grown' forests of planar rooted trees with the CM weightings giving the multiplicity of each type of tree and, therefore, each distinct ED. Later Merson and Butcher, in the 1950s and 60s, independently rediscovered this bijection in developing integration methods of the Runge-Kutta type to solve nonlinear differential equations. Butcher emphasized the group property of composition in his methods. This is discussed by K & M just after they introduce a flow equation and the CM elementary differentials on p. 14 (see also the Butcher group). See MO-Q link above for the meaning of 'naturally grown' and my relevant answer to the MO-Q "In splendid isolation" for more on the historical aspects and other approaches by other researchers.

Menous and Patras, in "Right-handed Hopf algebras and the preLie forest formula" in Sec. 4 The PreLie Forest Formula (pp. 10-12), illustrate precisely the LIF polynomials that Schröder was led to, the antipode, a.k.a. a Zimmerman forest formula (ZFF), of the combinatorial Faa di Bruno Hopf algebra. The other common antipode that pops up in QFT is the LIF of A133437 for the inverse o.g.f. of an o.g.f., related to the combinatorics of the faces of the associahedra (more precisely, the Euler characteristic partition polynomials of the associahedra).

The interplay of composition and decomposition among differentials and rooted trees formalized via Hopf algebras has been used by Broadhurst, Brouder, Connes, Kreimer, and others to remove divergences and calculate higher order summations of Feynman diagrams.

K & M give fairly enlightening discussions, toy models, and other examples of the correspondences sketched at the top.

M & P state:

These antipode formulas have been investigated by J.C. Figueroa and J.M. Gracia-Bondia [...] in the 2000s. They obtained a simple direct proof of Zimmermann's formula in QFT and showed more generally that one can employ the distributive lattice of order ideals associated with a general partially ordered set and incidence algebra techniques in order to resolve the combinatorics of overlapping divergences that motivated the development of the renormalization techniques of Bogoliubov, Dyson, Salam, Zimmermann et al.

Zeidler in sections 3.4.4 and 3.4.5, pp. 135-140, of "Quantum Field Theory II; Quantum Electrodynamics" has a leisurely discussion of the antipode / ZFF for the Faa di Bruno Hopf algebra, and, presents throughout the book, the relation of ZFFs to renormalization in QFT. Pages XI and XII of the Preface give a very nice intro to the topic.

Herzog, in "Zimmermann's forest formula, infrared divergences, and the QCD beta function", presents the beta function (eqn. 14 on p.5) in QCD up to five loops.

A common theme in analytic and algebraic geometry, dynamical and moduli spaces, characteristic and other special polynomials, combinatorics, and physics is the relationship between compositional and multiplicative inverse pairs, as might be expected since they are fundamental group operations (e.g., see refs for A145271, A133437, A134264, and A133314).

See also

- "[Hopf-algebraic renormalization of Kreimer's toy model]" by Panzer
- The brief overview "[Lessons from Quantum Field Theory -- Hopf Algebras and Spacetime Geometries]" by Connes and Kreimer.
- And hot off the press, "[Gentle introduction to rigorous Renormalization Group: a worked fermionic example]" by Giuliani, Mastropietroc, and Rychkovd

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Appendix 1: Peanuts and finding zero--on Schröder's functional series

In light of the analysis above, equation (21) in Schröder's paper cited above, using iterative derivatives to determine a zero of a function, becomes, with $g(z) = 1/f'(z)$,

$$z_1 = \sum_{n\geq 0} \frac{(-f)^n}{n!} \; (\frac{1}{f'} \; \partial_z)^n \; z = \sum_{n\geq 0} \frac{(-f(z))^n}{n!} \; (g(z) \; \partial_z)^n \; z$$

$$= Flow(t,z) \mid_{x=-f(z)} \Leftrightarrow F(x,z) \mid_{x=-f(z)}$$

$$= f_M^{(-1)}(x + f(z)) \mid_{x=-f(z)} = f_M^{(-1)}(0),$$

where, the arrow indicates a mapping via analytic continuation of the series typically to multiple solutions depending on the region in which z lies. As illustrated in the examples below, a non-bijective analytic function does not have a global inverse function but rather has different local inverse functions varying with the regions of z over which different bijections locally hold. The subscript M indexes these multiple local inverse functions. Then $z_1 = f_M^{(-1)}(0)$ implies $f(z_1) = f(f_M^{(-1)}(0)) = 0$ and we have our zeros.

Example I:

$f(z) = (1-z)^2$ and $f_{\pm}^{(-1)}(z) = 1 \pm \sqrt{z}$.

Then the naive flow function is

$F_\pm(t,z) = f_\pm^{(-1)}(t+f(z)) = 1 \pm \sqrt{t+(1-z)^2}$.

This doesn't indicate which sign should be chosen for specific values of z and t to obtain smooth flow.

The actions of the IG are given by

$\frac{1}{f'(z)} = g(z) = -\frac{1}{2}\frac{1}{1-z}$,

$(\frac{1}{f'(z)}\ D)^0\ z = z$,

$(\frac{1}{f'(z)}\ D)\ z = -\frac{1}{2}\frac{1}{1-z}\ D\ z = -\frac{1}{2}\frac{1}{1-z}$,

and, for $n \geq 2$,

$$(g(z)D)^n\ z = (\frac{1}{f'(z)}D)^n\ z = (-\frac{1}{2}\frac{1}{1-z}\ D)^n\ z = (-1)^n\ \frac{1}{2^n}\ \frac{(2n-3)!!}{(1-z)^{2n-1}}.$$

The Taylor series for F is

$$F_\pm(t,z) = 1 \pm \sqrt{t+(1-z)^2}$$

$$= 1 \pm |1-z|\ \sqrt{1+\frac{t}{(1-z)^2}}$$

$$= 1 \pm |1-z| \pm \frac{1}{2}\frac{1}{|1-z|}\ t \pm \sum_{n\geq 2} (-1)^n\ \frac{1}{2^n}\ \frac{(2n-3)!!}{|1-z|^{2n-1}}\ \frac{t^n}{n!},$$

which is convergent for $|\frac{t}{(1-z)^2}| < 1$, whereas the exponentiated IG, analytically continued, gives

$$IGS(t,z) = \exp[\ t\ (g(z)\ D_z)^n\ ]\ z = \exp[\ t\ (\frac{1}{f'(z)} D_z)^n\ ]\ z$$

$$= z + \sum_{n \geq 1} (-1)^n\ \frac{1}{2^n}\ \frac{(2n-3)!!}{(1-z)^{2n-1}}\ \frac{t^n}{n!}$$

$$= 1 - (1-z)\ \sqrt{1 + \frac{t}{(1-z)^2}}$$

$$= F_{-sign(1-z)}(t,z) = Flow(t,z).$$

When convergent, for $z > 1$, the series gives $F_+(t,z))$, and for $z < 1$, $F_-(t,z)$. The derivative $f'(z) = 2(1-z)$ changes sign as z passes through 1. The series can be expressed also as

$$\exp[\ t\ (\frac{1}{f'(z)} D_z)^n\ ]\ z = z + \sum_{n \geq 1} (-1)^n\ 2^{n-1}\ \frac{(2n-3)!!}{(f'(z))^{2n-1}}\ \frac{t^n}{n!}.$$

Approximating with $n = 100$ gives $F_-(.5,.2) \simeq -.0677078 \simeq IGS(.5,.2)$ and $F_+(.5,.2) = (x_1 + x_2) - F_-(.2,.5) \simeq 2.0677808.$

Approximating with $n = 100$ gives $F_+(.5,4) \simeq 4.082207 \simeq IGS(.5,4)$ and $F_-(.5,4) = 2 - F_+(.5,4) \simeq -2.082207$.

For numerical investigation, examples for the flow series and function for use with Wolfram Alpha are

.2 - (1/2) (1/(1-.2)) t + sum ((-1)^n (1/2^n) (2n-3)!! (1/(1-.2)^(2n-1))) (t^n/n!) , n=2, 100

and

1+ sqrt(.5 +(1-.2)^2).

Now what will Schröder’s formula give?

$$Flow(-f(z), z) = f^{(-1)}_{-sign(f'(z))}(-f(z) + f(z))$$

$$= f^{(-1)}_{-sign(f'(z))}(0) = \exp[\ t\ (\frac{1}{f'(z)} D_z)^n\ ]\ z\ |_{t=-f(z)}$$

$$= z - \frac{1}{2}\frac{1}{1-z}\ (-(1-z)^2) + \sum_{n\geq 2} (-1)^n\ \frac{1}{2^n}\ \frac{(2n-3)!!}{(1-z)^{2n-1}}\ \frac{(-(1-z)^2)^n}{n!}.$$

$$= z + \frac{1}{2}(1-z) + \sum_{n\geq 2}\ \frac{1}{2^n}\ (2n-3)!!\ (1-z)\ \frac{1}{n!}$$

$$= z + \frac{(1-z)}{2} + (1-z)\sum_{n\geq 2}\ \frac{(2n-3)!!}{2^n\ n!}$$

$$= 1 - (1-z) + \frac{(1-z)}{2} + (1-z)\sum_{n\geq 2}\ \frac{(2n-3)!!}{2^n\ n!}$$

$$= 1 + (1-z)\ [\ -\frac{1}{2} + \sum_{n\geq 2}\ \frac{(2n-3)!!}{2^n\ n!}\ ] = 1$$

since as can be shown from the more general case below

$\sum_{n\geq 2}\ \frac{(2n-3)!!}{2^n\ n!} = .5$, giving the double zero of $y = f(z) = (1-z)^2$.

Example II: A general quadratic

$$f(z) = (z_1 - z)(z_2 - z) = z^2 - (z_1 + z_2)z + z_1 z_2$$

$$= z^2 + e_1(z_1, z_2)\ z + e_2(z_1, z_2) = z^2 + bz + c,$$

$$f_{\pm}^{(-1)}(z) = \frac{(z_1 + z_2) \pm \sqrt{(z_1 + z_2)^2 - 4(z_1 z_2 - z)}}{2} =$$

$$= \frac{-b \pm \sqrt{b^2 - 4(c - z)}}{2} = \frac{-b \pm \sqrt{(b^2 - 4c) + 4z}}{2},$$

$$f'(z) = 2z - (z_1 + z_2) = 2z - e_1(z_1, z_2) = 2z + b,$$

$$\frac{1}{f'(z)} = g(z) = \frac{1}{2z + b},$$

$$(\frac{1}{f'(z)} D_z)^0\ z = z,$$

$$(\frac{1}{f'(z)}\ D_z)\ z = \frac{1}{2z + b}\ D\ z = \frac{1}{2z + b},$$

and, for $n \geq 2$,

$$(\frac{1}{f'(z)} D)^n\ z = (\frac{1}{2z + b} D)^n\ z = (-1)^{n+1}\ 2^{n-1}\ \frac{(2n - 3)!!}{(2z + b)^{2n-1}}.$$

The 'naive' flow function is

$$F_{\pm}(t, z) = f_{\pm}^{(-1)}(t + f(z))$$

$$= \frac{-b \pm \sqrt{b^2 - 4(c - (t + z^2 + bz + c))}}{2} = \frac{-b \pm \sqrt{b^2 + 4(z^2 + bz + t)}}{2}$$

$$= \frac{-b \pm \sqrt{(2z + b)^2 + 4t)}}{2}$$

the zeros of

$y(x) = x^2 + bx - (z^2 + bz + t)$

and the zeros shifted by z of

$y(x) = x^2 + (2z + b)x - t$

with

$$F_{\pm}(t,0) = f_{\pm}^{(-1)}(t + f(0)) = \frac{-b \pm \sqrt{b^2 + 4t}}{2}$$

and

$$F_{\pm}(0,z) = f_{\pm}^{(-1)}(f(z)) = \frac{-b \pm \sqrt{b^2 + 4(z^2 + bz)}}{2}$$

$$= \frac{-b \pm \sqrt{4\ (z + \frac{b}{2})^2}}{2} = -\frac{b}{2} \pm (z + \frac{b}{2}) = z\ or\ -(z+b).$$

Consistently,

$$F_{\pm}(0,0) = f_{\pm}^{(-1)}(f(0)) = f_{\pm}^{(-1)}(c) = \frac{-b \pm \sqrt{b^2}}{2} = 0\ or - b.$$

Checking the basic compositional iteration behavior--the translation property:

$$F_{\pm}(u, F_{u,\pm}(v,z)) = f_{\pm}^{(-1)}(u + f(f_{\pm}^{(-1)}(v + f(z))))$$

$$= f_{\pm}^{(-1)}(u + v + f(z)) = F_{\pm}(u+v, z)$$

$$= \frac{-b \pm \sqrt{(2(\frac{-b \pm \sqrt{(2z+b)^2 + 4v)}}{2}) + b)^2 + 4u)}}{2}$$

$$= \frac{-b \pm \sqrt{(2z+b)^2 + 4(u+v))}}{2}.$$

The associated IIG series (IGS) is

$$IGS(t,x) = \exp[\ t\ (\frac{1}{f'(z)}\ D_z)^n\ ]\ z$$

$$= z + \sum_{n \geq 1} (-1)^{n+1}\ 2^{n-1}\ \frac{(2n-3)!!}{(2z+b)^{2n-1}}\ \frac{t^n}{n!}$$

$$= z + \sum_{n \geq 1} (-1)^{n+1}\ Cat_n\ \frac{1}{(2z+b)^{2n-1}}\ t^n,$$

where $(2n-3)!!$ are the double factorials of A001147, with numerous combinatorial interpretations, and Cat_n are the even more chameleon-like Catalan numbers

$$(Cat_0, Cat_1,\ Cat_2, \ldots) = (0, 1,\ 1,\ 2,\ 5,\ 14,\ 42, \ldots)$$

of A000108, which have the o.g.f.

$$\frac{1}{1 - Cat.t} = \frac{1 - \sqrt{1-4t}}{2} = t + t^2 + 2t^3 + 5t^4 + 14t^5 + \ldots.$$

(The Catalan numbers occur as the number of vertices of the associahedra, the number of trees in the forests of rooted planar trees, the number of planted rooted trees in these forests, and, fittingly, as the solution to Schröder's first problem in his paper on four combinatorial problems, among a long list of other manifestations.)

Then

$$\sum_{n \geq 1} (-1)^{n+1}\ Cat_n\ \frac{1}{(2z+b)^{2n-1}}\ t^n = -(2z+b)\ \frac{1 - \sqrt{1 + 4\frac{t}{(2z+b)^2}}}{2}$$

$$= \frac{-(2z+b) + sign(2z+b)\ \sqrt{(2z+b)^2 + 4t}}{2}$$

$$= \frac{-(2z+b) + sign(2z+b)\ \sqrt{b^2 + 4(z^2 + bz + t))}}{2}$$

$$= \frac{-\bar{b} + sign(\bar{b})\ \sqrt{\bar{b}^2 - 4\bar{c}}}{2}$$

with $\bar{b} = 2z + b = f'(z)$ and $\bar{c} = -t$, the ancient venerable quadratic formula, and, for $|\frac{t}{(2z+b)^2}| < \frac{1}{4}$,

$$IGS(t,z) = \exp[\ t\ (g(z)\ D_z)^n\ ]\ z = \exp[\ t\ (\frac{1}{f'(z)}\ D_z)^n\ ]\ z$$

$$= \frac{-b + sign(2z+b)\sqrt{b^2 + 4(t + z^2 + bz))}}{2}$$

$$= \frac{-b + sign(2z+b)\sqrt{(2z+b)^2 + 4t}}{2}$$

$$= z + \frac{-\bar{b} + sign(\bar{b})\ \sqrt{\bar{b}^2 - 4\bar{c}}}{2}$$

$$= z + \frac{-\bar{b} + sign(f'(z))\ \sqrt{\bar{b}^2 - 4\bar{c}}}{2}$$

$$= F_{sign(2z+b)}(t,z) = F_{sign(f'(z))}(t,z)$$

$$= Flow(t,z).$$

Note $F_-(t,z) = -b - F_+(t,z).$

Note also

$f'(z) = 2z + b$ and

$x_1 + x_2 = -b$, so

$\frac{1}{f'(x_1)} = \frac{1}{2x_1 + b}$ and

$$\frac{1}{f'(x_2)} = \frac{1}{2x_2 + b} = \frac{1}{-2(x_1 + b) + b} = -\frac{1}{2x_1 + b} = -\frac{1}{f'(x_1)}.$$

.

Then with $t = -f(z) = -(z^2 + b\ z + c)$, we have the Theremin-Schröder series

$$z_1 = IGS(-f(z), z) = Flow(-f(z), z)$$

$$= \frac{-b + sign(2z + b)\ \sqrt{b^2 + 4(-(z^2 + bz + c) + z^2 + bz)}}{2}$$

$$= \frac{-b + sign(2z + b)\ \sqrt{b^2 - 4c}}{2}$$

$$= \frac{-b + sign(f'(z))\ \sqrt{b^2 - 4c}}{2}$$

$$= f^{(-1)}_{sign(-2f(z)+b)}(0)$$

which gives the two roots of $f(z) = x^2 + bx + c = 0$ for real zeros--the larger one when $f'(z) = 2z + b > 0$, or $z > -b/2 = (z_+ + z_-)/2$, i.e.,when z is to the right of the minimum of the quadratic and the smaller when z is to the left. $z = -b/2$ returns $z_1 = -b/2$, the z -coordinate of the minimum, for the analytic continuation, but the series itself is divergent.

$Flow(t, z)$ for the quadratic case is the sum of z and a zero of the quadratic equation $y(x) = F(x) = x^2 + \bar{b}x + \bar{c} = (x - x_1)(x - x_2)$ with $\bar{b} = f'(z) = 2z + b$ and $\bar{c} = -t$.

To get a feel for the effects of varying z and t on the quadratic curve $y = F(x)$, note completing the square gives

$$y = F(x) = x^2 + \bar{b}x + \bar{c} = (x - x_1)(x - x_2) = (x - h)^2 + v = Q(x - h) + v$$

with

$Q(x) = x^2$,

$h = -\frac{\bar{b}}{2} = \frac{x_1 + x_2}{2}$,

$v = \bar{c} - \frac{\bar{b}^2}{4} = -(\frac{x_1 - x_2}{2})^2$.

Then, for x_1 and x_2 real, $F(x)$ can be regarded as horizontal (h) and vertical v) translations of the basic quadratic curve $y = Q(x) = x^2$, quite reasonable since downward vertical translation affects only the distance between the zeros and not the x-coordinate of their midpoint and horizontal translation, vice versa. These are pure translations, so the orientation and shape of the curve $y = F(x)$ does not vary with changes in v and h from that of $y = Q(x)$. The x-coordinate of the midpoint between the real zeros of $F(x)$ is $m_x = -\frac{\bar{b}}{2}$ and this is also the x-coordinate of the minimum since $F'(x) = 2x + \bar{b}$. To understand the motions induced by changing z and t, we need consider only action on $y = Q(x) = x^2$. Varying t, and therefore $\bar{c}$, and adding this to $Q(x)$ displaces $y = Q(x)$ vertically to coincide with the curve $y = x^2 + \bar{c} = x^2 - t$ while varying z, and therefore $\bar{b}$, translates $y = F(x)$ both vertically and horizontally for adding $\bar{b}x$ to $y = Q(x)$ moves the minimum of $y = Q(x) = x^2$ at the origin along the curve $y = -x^2 = -Q(x)$ so that our initial quadratic $y = Q(x)$ now coincides with $y = \bar{Q}(x) = x^2 + \bar{b}x = x(x + \bar{b})$. The origin remains a zero of the curve while its minimum moves along $y = -Q(x)$. The effects of varying z or $\bar{b}$ follow analytically from the vanishing of $F'(x) = \bar{Q}'(x) = 2x + \bar{b} = 2x + (2z + b)$ at the minimum of $y = \bar{Q}(x)$, so the minimum of $y = \bar{Q}(x) = x^2 + \bar{b}x$ flows along the parametric curve $(x, y) = (-\bar{b}/2, -(\bar{b}/2)^2)$, i.e., along $y = -Q(x) = -x^2$. The motions induced by varying t and z, or $\bar{b}$ and $\bar{c}$, commute and linearly superpose on each other. (Use the Desmos graphing calculator with sliders for $\bar{b}$ and $\bar{c}$ to confirm visually these motions.). Another perspective for getting a handle on the effect of varying z is to note that $F'(0) = \bar{b} = 2z + b$ is the tangent to the quadratic at $(0, F(0)) = (0, \bar{c}) = (0, -t)$, so, for $t = 0$, varying z moves the quadratic $y = \bar{Q}(x)$ such that this is the tangent to the quadratic at the origin.

Numerical checks with Wolfram Alpha using

(-b + sqrt(b^2 + 4(t+z^2+bz)))/2

and

z + (1/(2z+b)) t + sum ((-1)^(n+1) (2^(n+1)) (2n-3)!! (1/(2z+b)^(2n-1))) (t^n/n!), n=2,100

give, for $b = 1$, $z = 0$, and $t = 1/8$,

(-1 + sqrt(1^2 + 4((1/8)+(0)^2+1(0))))/2 as approximately .112372

and

0+ (1/(2(0)+1)) (1/8) + sum ((-1)^(n+1) (2^(n-1)) (2n-3)!! (1/(2(0)+1)^(2n-1))) ((1/8)^n/n!), n=2,100 as approximately .112372.

__Appendix 2: Taylor series expansion of Schröder and Theremin__

Ignoring the intricacies of the lack of a unique inverse function for a non-bijective function, I've sketched below the arguments in Theremin and Schröder based on Taylor series machinations. The main features to take from this presentation are the roles of the general inversion polynomials and the fact that the two investigators arrive at an evaluation of a flow function though they do not couch their arguments in this language.

With $y = f(x)$ and $x = f^{(-1)}(y)$, suppressing arguments and using subscripts to indicate the order of differentiation,

$$\frac{d}{dy} = \frac{dx}{dy}\frac{d}{dx} = \frac{1}{f'(x)}\frac{d}{dx} = \frac{1}{f'}\frac{d}{dx} = \frac{1}{f_1}\frac{d}{dx},$$

and the actions of the first few iterated derivatives $\frac{d^n x}{dy^n} = (\frac{1}{f'(x)}\frac{d}{dx})^n\, x$ are

$$\frac{dx}{dy} = \frac{1}{f_1}\frac{d}{dx}\, x = \frac{1}{f_1},$$

$$\frac{d^2x}{dy^2} = \frac{1}{f'}\frac{d}{dx}\frac{1}{f'} = -\frac{f''}{(f')^3} = -\frac{f_2}{f_1^3},$$

$$\frac{d^3x}{dy^3} = \frac{1}{f'}\frac{d}{dx}(-\frac{f''}{(f')^3}) = \frac{3(f'')^2 - f'''f'}{(f')^5} = \frac{3f_2^2 - f_3 f_1}{f_1^5},$$

$$\frac{d^4x}{dy^4} = \frac{-15f_2^3 + 10f_1f_2f_3 - f_1^2f_4}{f_1^7},$$

$$\frac{d^5x}{dy^5} = \frac{105f_2^4 - 105f_1f_2^2f_3 + 10f_1^2f_3^2 + 15f_1^2f_2f_4 - f_1^3f_5}{f_1^9}.$$

These are the polynomials on p. 330 of Schröder and of the OEIS entry A134685. There are some transcription errors in Stewart's translation (p. 15). Theremin shows only up through the third order derivative in his 1855 paper.

Note the first derivative gives

$$\frac{dx(y)}{dy} = \frac{1}{f'(x)} = g(x(y)),$$

in general a nonlinear autonomous flow ODE for $x(y) = f^{(-1)}(y)$.

Schröder and Theremin expand the inverse function $x = f^{(-1)}(y)$ as a formal Taylor series about a zero of $y = f(x)$, say $y(x_0) = f(x_0) = 0$, as

$$x = f^{(-1)}(y)$$

$$= f^{(-1)}(y(x_0)) + (f^{(-1)})'(y(x_0))\ (y - y(x_0)) + (f^{(-1)})''(y(x_0))\ \tfrac{(y-y(x_0))}{2!} + \ldots$$

$$= f^{(-1)}(0) + (f^{(-1)})'(0)\ y + (f^{(-1)})''(0)\ \tfrac{y^2}{2!} + \ldots$$

$$= [u + \tfrac{dx(u)}{du}\ y + \tfrac{d^2x(u)}{du^2}\ \tfrac{y^2}{2!} + \ldots]\ |_{u=x_0}$$

$$= \exp(y\ \tfrac{1}{f'(u)}\partial_u)\ u\ |_{u=x_0}$$

$$= x_0 + \frac{1}{f_1(x_0)}\ y(x) + \frac{-f_2(x_0)}{f_1^3(x_0)}\ \frac{y^2(x)}{2!} + \frac{3f_2^2(x_0) - f_3(x_0)f_1(x_0)}{f_1^5(x_0)}\ \frac{y^3(x)}{3!} + \ldots$$

$$= x_0 + \frac{1}{f_1(x_0)}\ f(x) + \frac{-f_2(x_0)}{f_1^3(x_0)}\ \frac{f^2(x)}{2!} + \frac{3f_2^2(x_0) - f_3(x_0)f_1(x_0)}{f_1^5(x_0)}\ \frac{f^3(x)}{3!} + \ldots.$$

With the initial x replaced by z_1, x_0 by z, and $f(x)$ by $-f(z)$, this becomes equation 21 on p. 330 of Schröder

$$z_1 = z + \frac{1}{f_1(z)}\ (-f(z)) + \frac{-f_2(z)}{f_1^3(z)}\ \frac{(-f(z))^2}{2!} + \frac{3f_2^2(z) - f_3(z)f_1(z)}{f_1^5(z)}\ \frac{(-f(z))^3}{3!} + \ldots.$$

As I have shown above, when convergent,

$$x = \exp[y\ \frac{1}{f'(u)}\partial_u]\ u\ |_{u=x_0} = f^{(-1)}(y + f(u))\ |_{u=x_0, y=f(x)} = f^{(-1)}(f(x) + f(x_0)),$$

and performing the substitutions, Schröder's series equation becomes, when convergent,

$$z_1 = \exp[(y \, \frac{1}{f'(u)} \partial_u] \; u \; |_{u=z, \; y=-f(z)} = f^{(-1)}(y + f(z)) \; |_{y=-f(z)}$$

$$= f^{(-1)}(-f(z) + f(z)) = f^{(-1)}(0) = x_0,$$

a zero of $f(z)$.

With the series truncated at order (m-1), this is eqn. (17) on pg. 1759 of “On Schroder’s families of root-finding methods” by M. Petovic, L. Petovic, and D. Herceg. Examples are given beneath eqn. (19) on pg. 1760 in which the inversion polynomials A133437 occur, related to the associahedra. According to the authors, the equation is attributed in the Russian literature to Chebyshev (1837 or 1838), but others ascribe it to Euler.

Appendix 3: Schröder and missed opportunities

Apparently because Schröder was eager to develop an ‘absolute algebra’ based on mathematical logic, a phrase he coined, he missed an opportunity to scoop late 20’th century mathematicians on some early 21’st century mathematics. There’s been a renewed interest in the associations among the two families of convex polytopes the associahedra and permutahedra, Lagrange inversion formulas, and scattering processes in certain QFTs.

To set the stage: The last of the grand Druids (as Keynes would put it--after all it was an apple tree!), Newton, seems to be the first to have written out the first few inversion polynomials of normLIZED A133437 in a formula he developed for compositional inversion of power series (o.g.f.s), which turned out to be the refined Euler characteristic (or signed face) polynomials of the associahedra. Loday, circa 2000, seems to be one of the first to have pointed out this relationship between the associahedra and compositional inversion. In the 1820s we have Heinrich Scherk exploring infinigens of the Lie type and iterated derivatives and also Abel exploring functional equations that have the form of what are now called formal group laws, which are special cases of the flow function $F(t,y) = f^{(-1)}(t + f(y))$ presented above with $t = f(x)$. Cayley and the Graves brothers, John and Charles, come around in the 1850s to develop relationships among differential operators, flow functions, commutators, and combinatorial tree models along with the combinatorics of polygon dissections with Kirkman and much earlier with Segner, Fuss, Catalan, and Euler. An appetite for abstract algebra is whetted in 1843 by Hamilton’s invention of the quaternions and John Graves’, of the octonions and perhaps Hamilton’s development of the icosian calculus and its relation to moves on the vertices of a dodecahedron. And, we have well before all these illustrious figures, the mathemage Archimedes presenting the 3-dimensional version of the family of convex polytopes called the permutahedra / permutohedra.

Enter Schröder into this milieu with his penchant for combinatorics, associations with Klein and Hermann Grassmann, studies of Abel and Cayley, and desire to develop the fundamental foundations of an ‘absolute algebra’. In "On the exterior calculus and invariant theory," Barnabei, Brini, and Rota write, "To the best of our knowledge of published work, the first mathematicians to understand, albeit imperfectly, the program of [H. Grassmann's] Ausdehnungslehre were Clifford and Schröder ..." and "It was Schröder, in an appendix to his “Algebra der Logik,” who first stressed the analogy between the algebra of progressive and regressive products, and the algebra of sets with union and intersection." His interests were diverse, extending into physics and chemistry. Schröder, in his doctoral thesis in 1862, defined p/q-polygons, polygons with a fractional number of sides and, between 1869 and 1871, he published his paper “On four combinatorial problems” on the distribution, under different restrictions, of brackets, or parentheses, among strings of symbols, giving the Catalan numbers A000108, the little (A001003, Wikipedia) and large (A006318, Wiki) Schröder numbers, and A000311.

As I mentioned above in the analysis of his work on FI in extending Newton’s, Raphson’s, and Lagrange’s numerical methods for finding the zeros of polynomials, Schröder generated the first few polynomials of A134685 for the coefficients of the formal Taylor series of the compositional inverse of a formal Taylor series (he cites Theremin, “Recherches sur la résolution des équations de tous les dégres” as an influence, 1855). This family of inversion polynomials can be rescaled to give the power series (o.g.f.) of the inverse of a power series, which would give Newton’s inversion polynomials and the combinatorics of the associahedra (which had not yet been invented at that time). In the same paper, Schröder generates the logarithmic polynomials A263634, a.k.a. the cumulant expansion polynomials of A127671, which are important in the theory of symmetric functions and the operator calculus of Appell Sheffer polynomials and contain the refined Euler characteristic / face polynomials of the permutahedra, A133314, which double as the polynomials for the formal Taylor series of the multiplicative inverse of a formal Taylor series, and, therefore, the polynomials for the coefficients of the infinigen $g(x)\ \partial = (1/f'(x))\ \partial$ for Schröder’s polynomials. He did not normalize these and perhaps for that reason did not recognize the number of distinct faces of the 2-D (a hexagon) and 3-D permutahedra (an Archimedean truncated octahedron) in the coefficients of the polynomials--a little surprising since his thesis was concerned with polygons, but not too surprising for I know of principle researchers even in the last five years who had not been aware of the connections of the polynomials they encountered in their work in QFT to those for the associahedra until I informed them--the perils of compartmentalization! I’m sure Schröder would have been thrilled to have noticed this, and it would perhaps have accelerated research on the interplay of geometry, differential operators, combinatorics, and algebra by several decades, maybe a century or more. He was poised to follow an enlightened path yet chose to leap into the abyss of mathematical logic, but he had his reasons and his fun.

Appendix 4: Maupertuis action (abbreviated / reduced / symplectic) for the harmonic oscillator

Reprising,

$$\frac{\partial S}{\partial E} = \int \frac{1}{\sqrt{\frac{2}{m}\,(E - U(x))}}\,dx - t$$
$$= \int \frac{1}{v(x)}\,dx - t,$$

and the infinigen for the HO is

$$g(x) = \beta(x) = v(x) = \sqrt{\frac{2}{m}\,(E - U(x))} = \sqrt{\frac{k}{m}x_0^2\,(1 - (\frac{x}{x_0})^2)}.$$

Scaling to dimensionless quantities, we have the infinigen

$$g(x) = \sqrt{1 - x^2}.$$

Then

$$h(x) = \int \frac{1}{\sqrt{1 - x^2}}\,dx = \arcsin x + C,$$

the inverse function is

$$h^{(-1)}(x) = \sin(x),$$

the naive flow function is

$$F(t, x) = h^{(-1)}(x)(t + h(x)) = \sin(t + \arcsin x),$$

and the associated series IGS is

$$Flow(t,x) = \exp[\ t\ g(x)\ \partial_x\ ]x = \exp[\ t\ \sqrt{\ 1-x^2}\ \partial_x\ ]x$$

$$= \sum_{n\geq 0} \frac{t^n}{n!}\ (g(x)\ \partial_x)^n x = \sum_{n\geq 0} \frac{t^n}{n!}(\sqrt{1-x^2}\ \partial_x)^n x.$$

The action of powers of the infinigen on x generates the inversion polynomials of A145271 in terms of $g(x)$ and its derivatives of. For our SHO, the actions are

$$(g(x)\partial_x)^0\ x = (\sqrt{1-x^2}\ \partial_x)^n\ x = x,$$

$$(g(x)\partial_x)^1\ x = (\sqrt{1-x^2}\ \partial_x)\ x = \sqrt{1-x^2},$$

$$(g(x)\partial_x)^2\ x = (\sqrt{1-x^2}\ \partial_x)\ x = \sqrt{1-x^2}\ \frac{1}{2}\ \frac{-2x}{\sqrt{1-x^2}} = -x,$$

and the pattern repeats, giving

$$IGS(t,x) = x + \sqrt{1-x^2}\ t - x\ \frac{t^2}{2!} - \sqrt{1-x^2}\ \frac{t^3}{3!} + \ldots = x\ \cos(t) + \sqrt{1-x^2}\ \sin(t).$$

Using Desmos with t and x as sliders, for $|x| \leq 1$, gives a visualization and numerical agreement of the equality

$$y(x) = x\ \cos(t) + \sqrt{1-x^2}\ \sin(t) = \sin(t + \arcsin(x))$$

as a variable is changed.Sliding t gives the projection onto the plane of a half-circle rotating on the sphere $r = 1$ centered about the origin. For $t = 2\pi$, the projection is the line $y = x$, and for $t = \pi/2$, a half-circle above the x-axis on the plane.

Define the conjugate function

$$\bar{y}(x) = x\ \cos(s) + \sqrt{1-x^2}\ \sin(s) = \sin(s + \arcsin(x)).$$

Then when $s + t = 2\pi$, the two separate curves form a closed curve, each curve being a reflection of the other through $y = -x$.

x is constrained to lie between 1 and -1, so let $x = \cos(\theta)$ and $t = \phi$ where θ is the angle from the positive vertical z-axis and ϕ that for the x-axis. Then

$$F(t, x) = P(\phi, \theta) = \cos(\theta)\cos(\phi) + \sin(\theta)\sin(\phi) = \cos(\theta - \phi).$$

parametrizes the point (t, x) in spherical coordinates as a point moving on a sphere of unit radius.

Reprising,

$$S = \int \sqrt{2m\ (E - V(x))}\ dx - E\ t$$

$$= \int \sqrt{2m\ (\frac{1}{2}k\ x_0^2 - \frac{1}{2}k\ x^2)}\ dx - \frac{1}{2}k\ x_0^2\ t$$

$$= \sqrt{m\ k}\ x_0 \int \sqrt{\ 1 - (\frac{x}{x_0})^2}\ dx - \frac{1}{2}k\ x_0^2\ t$$

$$= \frac{1}{2}\ \sqrt{m\ k}\ x_0\ [\ x\sqrt{1 - (\frac{x}{x_0})^2} + x_0\ \ \arcsin(\frac{x}{x_0}) + C\ ] - \frac{1}{2}k\ x_0^2\ t.$$

$$= \frac{m}{2}\ \sqrt{\frac{k}{m}}\ x_0^2\ [\ \cos(\sqrt{\frac{k}{m}}\ t)\ \ \sin(\sqrt{\frac{k}{m}}\ t) +\ \ \arcsin(\cos(\sqrt{\frac{k}{m}}\ t)) + C\ ] - \frac{1}{2}k\ x_0^2\ t$$

$$= \frac{m}{2}\ \omega\ x_0^2\ [\ \cos(\omega\ t)\ \ \sin(\omega\ t) +\ \ \arcsin(\cos(\omega\ t)) + C - \omega\ \ t\ ].$$

(Aside from the last energy term linear in the time and an overall scale factor, this is equation (3) in "RG flows, cycles, and c-theorem folklore" by Curtright, Jin, and Zachos.)

Should have

$$\frac{\partial S}{\partial x} = p = \sqrt{2m\ (E - U(x))} = \sqrt{2m\ (\frac{1}{2}k\ x_0^2 - \frac{1}{2}k\ x^2)}$$

and

$$\frac{\partial S}{\partial x} = \frac{1}{2}\sqrt{m\,k}\,x_0\,\frac{\partial}{\partial x}\,[\,x\sqrt{1-(\frac{x}{x_0})^2}+x_0\ \arcsin(\frac{x}{x_0})\,].$$

Series expansion gives

$$\frac{1}{2}[\,x\sqrt{1-(\frac{x}{x_0})^2}+x_0\ \arcsin(\frac{x}{x_0})\,]$$

$$= x - \frac{1}{6}\frac{x^3}{x_0^2} - \frac{1}{40}\frac{x^5}{x_0^4} - \frac{1}{112}\frac{x^7}{x_0^6} - \frac{5}{1152}\frac{x^9}{x_0^8} - \frac{7}{2816}\frac{x^{11}}{x_0^{10}} - \frac{21}{13312}\frac{x^{13}}{x_0^{12}} - \frac{11}{10240}\frac{x^{15}}{x_0^{14}}$$

$$-\frac{429}{557056}\frac{x^{17}}{x_0^{16}} - \ldots$$

$$= x_0(\frac{x}{x_0} - \frac{(\frac{x}{x_0})^3}{3!} - 3\frac{(\frac{x}{x_0})^5}{5!} - 45\frac{(\frac{x}{x_0})^7}{7!} - 1575\frac{(\frac{x}{x_0})^9}{9!} - 99225\frac{(\frac{x}{x_0})^{11}}{11!} - \ldots)$$

Some numerology:

The denominators are A002595 = 1, 6, 40, 112, 1152, 2816, 13312, 10240, 557056, 1245184,..., coefficients of the Taylor series expansion of $\arcsin(x)$. Also arises from arccos(x), arccsc(x), arcsec(x), arcsinh(x).

The numerators are A091154 = 1, 1, -1, 1, -5, 7, -21, 11, -429, 715, -2431, 4199, -29393 .... mod signs, the numerator of Taylor-Maclaurin expansion of the arc length of Archimedes' spiral, in polar coordinates

$$arclength = r\ \frac{1}{2}\ (x\ \sqrt{1+\theta^2} + \sinh^{(-1)}(\theta))/2$$

$$= r\ \frac{1}{2}\ [\ ]x\ \sqrt{1+\theta^2} + \ln(\theta + \sqrt{1+\theta^2})\ ]$$

$$= r[\theta + \frac{1}{2}\sum_{n=3}^{\infty} [\ P_{n-3}(0) + \frac{n+1}{n} P_{n-1}(0)]\ \theta^n\ ]$$

$$= r[\ \theta + \frac{\theta^3}{6} - \frac{\theta^5}{40} + \frac{\theta^7}{112} - \cdots].$$

The Taylor series coefficients mod-signs are A079484 = 1, 3, 45, 1575, 99225, 9823275, 1404728325, ... , $a(n) = \frac{(2n-1)!!}{(2n+1)!!}$, where the double factorial is A006882 = 1, 2, 3, 8, 15, 48, 105, 384, 945, 3840, 10395, 46080, ..., containing the odd double factorials , defined recursively as $n!! =: n \cdot (n-2)!!$ for $n > 1$.
45/3 = 15, 315/45 = 7, 14175/315 = 45, 467775/14175 = 33, 42567525/ 467775 = 91,

Divide the coefficients by A117972 =1, -1, 3, -45, 315, -14175, 467775, -42567525, 638512875, -97692469875, 9280784638125 , the numerators of $zeta'(-2n)$ (the derivative of zeta) to get the integer sequence

A079484 / A117972 =

1/1, 3/3, 45/45, 1575/315, 99225/14175, 9823275/467775, 1404728325/42567525, 273922023375/638512875, 69850115960625/97692469875, 22561587455281875/9280784638125

https://oeis.org/A002596 = 1, 1, -1, 1, -5, 7, -21, 33, -429, 715, -2431, 4199, … , the numerators in the expansion of $\sqrt{1+x}$. The absolute values give the numerators in the expansion of $\sqrt{1-x}$, also the numerators of $\frac{(2n-3)!!}{n!}$ or the odd part of the $(n-1)$-th Catalan number. . Also in the “Addendum to the Elliptic Lie Triad.” Also A098597: 1, 1, 1, 5, 7, 21, 33, 429, 715, 2431, 4199, 29393, 52003, 185725, … , the numerators of $\frac{Catalan(n)}{2^{2n+1}}$, the odd part of the n-th Catalan number, also the numerators of $(2n-1)!!/(n+1)$.?????? Check these ????? Essentially the last two sequences are the same except for the initial values. They appear in Addendum to the Lie Triad as well as numerators of series expansion of the two zeros of a

Riccati equation. See also “Combinatorial Identities Associated with a Multidimensional Polynomial Sequence” by Cacao and Malonek.

A117972 also A048896(n), n >= 1: Numerators of Maclaurin series for 1 - ((sin x)/x)^2,
a(n), n >= 2: Denominators of Maclaurin series for 1 - ((sin x)/x)^2, the correlation function in Montgomery's pair correlation conjecture.

-1/4, 3/4, -45/8, 315/4, -14175/8, 467775/8, -42567525/16, ...
-zeta(3)/(4*Pi^2), (3*zeta(5))/(4*Pi^4), (-45*zeta(7))/(8*Pi^6), (315*zeta(9))/(4*Pi^8), (-14175*zeta(11))/(8*Pi^10), ...

From MathWorld,

$$\zeta'(-2n) = (-1)^n \ \frac{\zeta(2n+1)(2n)!}{2^{2n+1}\pi^{2n}}$$

$\zeta'(-2) = -\frac{\zeta(3)}{4\pi^2}$

$\zeta'(-4) = 3\frac{\zeta(5)}{4\pi^4}$

$\zeta'(-6) = -45\frac{\zeta(7)}{2\cdot 4\pi^6}$

$\zeta'(-8) = 315\frac{\zeta(9)}{4\pi^8}$

Cat_n: 1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, 208012, 742900,

1, 1, 2/2 = 1, 5, 14/2 =7, 42/2 = 21, 132/4 = 33, 429, 1430/2 = 715, 4862/2=2431, 16796/4 = 4199 , 58786/2=29393, 208012/4 = 52003, 742900/4 = 185725

Divisors 1,1,2,1,2,2,4,1,2,2,4,2,4,4 are https://oeis.org/A048896.

A098597: 1, 1, 1, 5, 7, 21, 33, 429, 715, 2431, 4199, 29393, 52003, 185725, … , the numerators of ,

$$4\pi^8 \frac{\zeta'(-8)}{\zeta(9)} \cdot N[\frac{Cat_3}{2^{2\cdot 3+1}}] =$$

The derivatives may be given by the reflection formula as derivatives at zeta(n) and also D_x \ln(\zeta) = \zeta’/zeta.

From my MO-Q,

$$\frac{2}{(2\pi)^{2n}} \ (2n-1)! \, \zeta(2n) = (-1)^{n+1} \frac{B_{2n}}{2n} = (-1)^n \zeta(1-2n)$$

For $n \geq 1$,

$$\zeta(2n) = (-1)^{n+1} \frac{(2\pi)^{2n}}{2(2n)!} B_{2n}.$$

Numerators of the Bernoulli numbers https://oeis.org/A027641 = 1, -1, 1, 0, -1, 0, 1, 0, -1, 0, 5, 0, -691, 0, 7, 0, -3617, 0, 43867, 0, -174611

$$\sqrt{1-x^2}$$

$$= 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - \frac{7}{256}x^{10} - \frac{21}{1024}x^{12} - \frac{33}{2048}x^{14} - \frac{429}{32768}x^{16} - \ldots$$

$$= 1 - \frac{1}{2}x^2 - \frac{1}{2^3}x^4 - \frac{1}{2^4}x^6 - \frac{5}{2^7}x^8 - \frac{7}{2^8}x^{10} - \frac{21}{2^{10}}x^{12} - \frac{33}{2^{11}}x^{14} - \frac{429}{2^{15}}x^{16} - \ldots$$

$$= 1 - \frac{x^2}{2!} - 3\frac{x^4}{4!} - 45\frac{x^6}{6!} - 1575\frac{x^8}{8!} - 99255\frac{x^{10}}{10!} - 9823275\frac{x^{12}}{12!} - \ldots$$

The Taylor series has the coefficients A079484 = 1,-1,-3,-45,-1575, -99255, … .

Powers of 2^n in the denominators are https://oeis.org/A005187 = 0, 1, 3, 4, 7, 8, 10, 11, 15, 16, 18, 19, 22, … , $a(n) = a(floor(n/2)) + n$, also denominators in expansion of $\frac{1}{\sqrt{1-x}}$ are $2^{a(n)}$, also $2n$ - number of 1's in binary expansion of $2n$, pops up in the Whitney immersion theorem.

Apply A145271 for the inverse:

Let $(j')^k = (D^j g(x))^k = (g^{(j)}(x))^k$.

$(0') = g(x) = \sqrt{1-x^2}$,

$(1') = D\ g(x) = \frac{-x}{\sqrt{1-x^2}}$,

$(2') = D^2\ g(x) = \frac{-1}{(1-x^2)^{3/2}}$,

$(3') = D^3\ g(x) = \frac{-3x}{(1-x^2)^{5/2}},$

$(4') = D^4\ g(x) = -3\frac{1+4x^2}{(1-x^2)^{7/2}},$

$(5') = D^5\ g(x) = -15x\frac{3+4x^2}{(1-x^2)^{9/2}},$

$(6') = D^6\ g(x) = -45\frac{1+12x^2+8x^4}{(1-x^2)^{11/2}}$

$(7') = D^7\ g(x) = -315\ x\ \frac{5+20x^2+8x^4}{(1-x^2)^{13/2}}$

$(8') = D^8\ g(x) = -315\ \frac{5+120x^2+240x^4+64x^6}{(1-x^2)^{15/2}}$

$(9') = D^9\ g(x) = -2835\ x\ \frac{35+280x^2+336x^4+64x^6}{(1-x^2)^{17/2}}$

$(10') = D^{10}\ g(x) = -14175\ \frac{7+280x^2+1120x^4+896x^6+128x^8}{(1-x^2)^{19/2}}$

$(11') = -155925\ x\ \frac{128x^8+1152x^6+2016x^4+840x^2+63}{(1-x^2)^{21/2}}.$

The numerator polynomials all have zeros on the imaginary axis. They are a normalized version of the numerator polynomials of A091894, which has as an o.g.f (expanding in t)

$f(x,t) = \frac{1-2t-\sqrt{(1-2t)^2-4xt^2}}{2tx}$

$= \frac{1-2t-\sqrt{1-4t-4(x-1)t^2}}{2tx}$

with the inverse in t

$f^{(-1)}(x,t) = \frac{t}{1+2t+xt^2} = \frac{xt}{(tx+1)^2+(x-1)}$

and are, therefore, related to the Legendre and Gegenbauer polynomials.

$f(x,t)$ gives a zero of

$z(y) = 2tx\ y^2 - (1-2t)\ y + \frac{t}{2},$

so perhaps can be related to the Riccati equation

$$y' = 2tx\ y^2 - (1-2t)\ y + \frac{t}{2}$$

and, therefore, the elliptic triad. The polynomials from this g.f. are not divided by any function and are not alternatingly even and odd polynomials.

In polar coordinates, with $y(x) = \sqrt{1-x^2} = \sin(\theta)$ and $x = \cos(\theta)$ for $0 \leq \theta \leq \pi$,

$$\frac{d^n}{dx^n}\sqrt{1-x^2} = (-\frac{1}{\sin(\theta)}\frac{d}{d\theta})^n \sin(\theta).$$

$$g(\theta) = -\frac{1}{\sin(\theta)} = \frac{1}{\cos'(\theta)}$$

$f(u) = \cos(u)$ and $f^{(-1)}(u) = \arccos(u)$

$$\exp[t(-\frac{1}{\sin(\theta)}\frac{d}{d\theta})]\ \sin(\theta) = \sin[\arccos(t+\cos(\theta)] = \sin[\arccos(t+x)],$$

so the derivatives also have the generating function

$$\sin[\arccos(t+x)]$$

but that should be the same as the series in t of

$$e^{tD_x}\ \sqrt{1-x^2} = \sqrt{1-(x+t)^2} = \sqrt{(1-t^2)-2tx-x^2}.$$

Same numerator polynomials mod signs generated by $\pm\sqrt{1\pm(x+t)^2}$.

Alpha Wolfram confirms.

Compare with

$$\frac{1-2t-\sqrt{(1-2t)^2-t^2 4x}}{2tx}$$

$$= \frac{1-2t-\sqrt{1-4t-4(x-1)t^2}}{2tx}$$

Wolfram Alpha gives

series in t of (1 - 2*t - sqrt((1 - 2*t)^2 - 4*t^2 *x)) / (2*t*x) gives the same as

series in t of ((1/(-x   *t))* \sqrt(1 +((1/(i*sqrt(x)))+(2*(i*sqrt(x))*(-t)*((-1/x) +1)))^2)/ \sqrt((-1/x) +1) /2   ) -(1/x)+(1/(2tx))

Inspection of the series expansion of $\sqrt{1+(x+t)^2}$ reveals that dividing the series by $\sqrt{1+x^2}$ and then replacing t by $2t(1+x^2)$ gives

$$\frac{\sqrt{1+(x+2t(1+x^2))^2}}{\sqrt{1+x^2}} = 1+2[xt+t^2-2xt^3+(4x^2-1)t^4-(8x^3-6x)t^5+\ldots].$$

Subtracting out the initial 1 and $2xt$, dividing out the 2, and changing signs leaves

$$\frac{1}{2}[1+2xt-\frac{\sqrt{1+(x+2t(1+x^2))^2}}{\sqrt{1+x^2}}] = -t^2+2xt^3-(4x^2-1)t^4+(8x^3-6x)t^5+\ldots.$$

Reversing the polynomials, halving their exponents, and removing the signs is achieved by replacing x by $\frac{1}{\sqrt{-x}} = \frac{1}{i\sqrt{x}}$ and the new t by $-t\ i\sqrt{x}$, giving the series

$$x\ (t^2+2t^3+(4+x)t^4+(8+6x)t^5+\ldots.)$$

Finally divide by tx. These machinations morph our initial g.f.

$$\sqrt{1-(x+t)^2}$$

into

$$\frac{1}{2tx}\,[1-2t-\frac{\sqrt{1+[\frac{1}{i\,\sqrt{x}}-(i\,\sqrt{x})\;2t(\frac{-1}{x}+1)]^2}}{\sqrt{\frac{-1}{x}+1}}]$$

$$=\frac{1}{2tx}\,[1-2t-\frac{\sqrt{1-\frac{1}{x}-4t(\frac{-1}{x}+1)-4t^2x(\frac{-1}{x}+1)^2}}{\sqrt{\frac{-1}{x}+1}}]$$

$$=\frac{1}{2tx}\,[\,1-2t-\sqrt{1-4t-4t^2(x-1)}\,],$$

which is an o.g.f. for the polynomials of A091894.

Wolfram Alpha check:

series in t of ((1/(-x *t))* \sqrt(1 +((1/(i*sqrt(x)))+(2*(i*sqrt(x))*(-t)*((-1/x) +1)))^2)/ \sqrt((-1/x) +1) /2) -(1/x)+(1/(2tx))

The factors normalizing the polynomials are signed A049606.

The coefficients of the lowest order terms of each numerator polynomial are

$3 \cdot 1 = 3$
$3 \cdot 1 = 3$
$15 \cdot 3 = 45$
$45 \cdot 1 = 45$
$315 \cdot 5 = 1575$
$315 \cdot 5 = 1575$
$2835 \cdot 35 = 14175 \cdot 7 = 99225$
$14175 \cdot 7 = 99225$
$155925 \cdot 63 = 9823275.$

The alternating number sequence is, of course, A079484 = 1, 3, 45, 1575, 99225, 9823275, 1404728325, 273922023375, ..., the unsigned Taylor series coefficients of $\sqrt{1-x^2}$.

Factors are https://oeis.org/A001790 = 1, 1, 3, 5, 35, 63, 231, 429 ?

1, 3, 45, 1575, 99225, 9823275, 1404728325, 273922023375, 69850115960625, 22561587455281875, 9002073394657468125, There are five OEIS entries for 3, 3, 15, 45, 315, 315, 2835, 14175, 155925. a(n+1)/a(n) =155925/14175 = 11 , 14175/2835 = 5, 2835/315 = 5, 315/315=1, 315/45 = 7, 45/15 = 3, 3/3 = 1 ????

https://oeis.org/A156769 and https://oeis.org/A036279 =1, 3, 15, 315, 2835, 155925, 6081075, 638512875, 10854718875, 1856156927625, 194896477400625 differ in next term.

Checking the formalism with the partition polynomials of A145271:

Let $R = g(x)\,\partial_x = \sqrt{1-x^2}\,\partial_x$ then

$R^0 g(x) = 1(0')^1 = \sqrt{1-x^2}$

$R^1 g(x) = 1(0')^1(1')^1 = \sqrt{1-x^2}\frac{-x}{\sqrt{1-x^2}} = -x$

$R^2 g(x) = 1(0')^1(1')^2 + 1(0')^2(2')^1$

$= \sqrt{1-x^2}(\frac{-x}{\sqrt{1-x^2}})^2 + (\sqrt{1-x^2})^2(\frac{-1}{(1-x^2)^{3/2}})$

$= \frac{x^2}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = -\sqrt{1-x^2}$

$R^3 g(x) = 1(0')^1(1')^3 + 4(0')^2(1')^1(2')^1 + 1(0')^3(3')^1$

$= \sqrt{1-x^2}(\frac{-x}{\sqrt{1-x^2}})^3 + 4(\sqrt{1-x^2})^2\frac{-x}{\sqrt{1-x^2}}\frac{-1}{(1-x^2)^{3/2}} + (\sqrt{1-x^2})^3\frac{-3x}{(1-x^2)^{5/2}}$

$= \frac{-x^3}{1-x^2} + \frac{4x}{1-x^2} - \frac{3x}{1-x^2} = x,$

in agreement with $f(x) = \arcsin(x)$, $g(x) = 1/f'(x) = \sqrt{1-x^2}$, $f^{(-1)}(x) = \arcsin(x)$ and the expansion of

$$\exp(t\ g(x)\ \partial_x)\ x = f^{(-1)}(t + f(x))$$

$$= \exp(t\ \sqrt{1-x^2}\ \partial_x)\ x = \sin(t + \arcsin(x)).$$

$R^{n-1}\ g(x)$ evaluated at $x = 0$ is the n-th Taylor series coefficient of the compositional inverse of $h(x) = \int_0^x\ 1/g(u)\ du$.

For some more recent notes on these relationships, see my last contribution (Oct. 2021) to my MO-Q “Geometric / physical / probabilistic interpretations of Riemann zeta(n>1)?”.

Appendix 5: Binomial Sheffer / Jabotinsky polynomials and functional iteration

Composition of functions $P(t)$ and $Q(t)$ with $P(0) = Q(0) = 0$ and nonvanishing first derivatives can be related to multiplication of matrices via the Sheffer calculus. In umbral notation, the binomial Sheffer polynomials $(p.(x))^n = p_n(x)$ and $(q.(x))^n = q_n(x)$ of degree n associated with these functions have the exponential generating functions (e.g.f.s)

$e^{t\ p.(x)} = e^{xP(t)}$ and $e^{t\ q.(x)} = e^{xQ(t)}$.

Let P and Q be the associated matrices of coefficients $p_{n,k}$ and $q_{n,k}$ of the polynomials. The matrix product

$$P \cdot Q = R$$

then represents the coefficient matrix with elements $r_{n,k}$ of the polynomials $r_n(x)$ resulting from the umbral composition

$$r_n(x) = p_n(q.(x)) = \sum_{k=0}^{n} p_{n.k}\ q_k(x).$$

Consequently, the e.g.f. for these resulting polynomials is

$$e^{t\ r.(x)} = e^{t\ p.(q.(x))} = e^{P(t)q.(x)} = e^{Q(P(t))x} = e^{R(t)x},$$

with

$$R(t) = Q(P(t)).$$

Repeated compositional / functional iteration of the function $Q(t)$, e.g., $Q^{((2))}(t) = Q(Q(t))$, can be represented as a power of the coefficient matrix Q--in the example Q^2, or equivalently, as the umbral composition $q_n(q.(x))$.

See also the refs on Jabotinsky and “Continuous Iteration of Dynamical Maps” by Aldrovandi and Freitas.

Appendix 6: Tangency / transport equation

In "Renormalization Group Functional Equations" by Curtright and Zachos, the authors give In their Appendix the traversal time for a particle moving in a conservative potential along a trajectory $x(t, x_0)$ from the initial position x_0 at $t = 0$ to $x(t, x_0)$ as (their eqn. (83)

$$\int_{x_0}^{x(t,x_0)} \frac{dx}{v(x)} = t.$$

They treat t and x_0 as independent variables and differentiating the equation w.r.t. x_0, they obtain the variational result

$$\frac{\partial x(t,x_0)}{\partial x_0} \, \frac{1}{v(x(t,x_0))} - \frac{1}{v(x_0)} = 0,$$

implying, since $v(x(t,x_0)) = \frac{\partial x(t,x_0)}{\partial t}$,

$$\frac{\partial x(t,x_0)}{\partial t} - v(x_0) \, \frac{\partial x(t,x_0)}{\partial x_0} = 0.$$

We can identify the flow function

$$x(t,x_0) = F(t,x_0) = x(t + x^{(-1)}(x_0)) = x(t + t_0),$$

and equation (89) of C & Z,

$$x(t,x_0) = \exp[t \; \partial_\tau] \; x(\tau, x_0) \; |_{x_0=0} = \exp[t \; v(x_0)\partial_{x_0}] \; x_0,$$

as the associated IIG series.

As C & Z point out, this may be put into a version of the Schröder FC equation by choosing a pair of inverse functions ψ and $\psi^{(-1)}$ with

$$v(x_0) = \frac{\psi(x_0)}{\psi'(x_0)} = \frac{1}{\partial_{x_0} \; \ln(\psi(x_0))}.$$

In the notation above, $f(x)$ corresponds to $\ln(\psi(x_0)$ and $f^{(-1)}(x)$ to $\psi^{(-1)}(e^{x_0})$, so substituting $\frac{1}{\partial_{x_0} \ \ln(\psi(x_0))}$ for $v(x_0)$ in the IIG series gives

$$x(t+t_0) = f^{(-1)}(t + f(x_0)) = \psi^{(-1)}(\exp t + \ln(\psi(x_0))) = \psi^{(-1)}(e^t \ \psi(x_0)).$$

Counterparts to these maneuvers can be found in equations (B. 14) - (B. 26) of one of the earliest papers (1954) on renormalization group flow "Quantum electrodynamics at small distances" by Gell-Mann and Low, which inspired Wilson to formulate the RG flow methodology. G & L thank the Nobel laureate T. D. Lee for suggesting this method of solution to their functional equations for photon propagation. As Shirkov points out in his contribution A.25 Functional Self-Similarity or Synthesis (starting on pg. 181 of Brown), the group transformations above are tantamount to similarity transformations, well-known by physicists, as discussed in the Rosenblatt and the Goldenfeld refs in the Related Stuff section below.

Another way to express this is

$$\exp[\ t \ z\partial_z \ ] \ \psi^{(-1)}(z) \ |_{z=\psi(x_0)} = \psi^{(-1)}(e^t \ \psi(x_0))$$

since $e^{t \ z \ \partial_z} \ w(z) = e^{t \ \partial_u} \ w(\ln u) = w(e^t \ z),$

with $z = e^u$, identified as the diff op / infinigen rep of the dilatation / dilation operator of SL_2. As C & Z point out, Sidney Coleman (a student of Gell-Mann) used the formalism above to to discuss renormalization in "Aspects of Symmetry" (in his 1971 lecture 3 “Dilatations, section 4 The resurrection of scale invariance, subsection 4.1 The renormalization group equations and their solution”, beginning on, coincidentally, the fortuitous p. 88), in which the two infinigens $\mu \ \partial_\mu$ and $\beta(\lambda) \ \partial_\lambda$ appear and an analogy is drawn to the density of a population of bacteria growing and moving with the flow of water in a pipe. (Compare this with the differential equation for $G(c, L)$ on p.76 of "A Combinatorial Perspective on Quantum Field Theory" by Yeats and the “Callan–Symanzik equation” on Wikipedia and on pg. 175 of “Renormalization From Lorentz to Landau (and Beyond)” compiled by Brown.) Coleman remarks, "The β anomalies, which complicate things terribly at low energies, simplify things enormously in the deep Euclidean region. What a wonderful reversal! (this praise of the β-terms is justified, of course, only if our critical assumption--the existence of a zero in β is justified. If it is not, the β-terms remain troublemakers, at high energies as well as low.)"

A more general transformation as in Coleman on pg. 77, $z \rightarrow \frac{az+z^2}{1+2az+z^2}$ with

$\frac{1}{2}\ z\partial_z\ \ln(1+2a\ z+z^2) = \frac{az+z^2}{1+2az+z^2}.$ Letting $z = e^u$, then $\frac{1}{2}\ \partial_u\ \ln(1+2ae^u+e^{2u}) = \frac{az+z^2}{1+2az+z^2}.$

Equation (11) on pg.366 of "On analytic iteration" by Erdos and Jabotinsky is a version of the tangency condition. Identifying their $L(z)$ with our $g(z)$. This is also eqn. (3.10) on pg. 463 of "Analytic iteration" by Jabotinsky.

Compare the general tangency PDE with equation (2) of "'Irreversibility' of the flux of the renormalization group in a 2D Field Theory" by Zamolodchikov and with the diffeq for the Loewner chain, a one-parameter family of conformal mappings, $\partial_t\ f_t(z) = z\ p_t(z)\ \partial_z\ f_t(z).$

The same basic ODE and tangency PDE are found in "The QCD β-function from global solutions to Dyson-Schwinger equations" by van Baalen, Kreimer, Uminsky, and Yeats (pg. 6, eqn. 4): The Dyson-Schwinger equation for γ_1 is $\frac{\partial\gamma_1(x)}{\partial x} = f(\gamma_1(x)).$ See pgs. 14 and 17 (eqns. 16 and 17) also for equations analogous, with obvious transformations, to the formalism above.

Eqn. 3.50 on pg. 74 of "A mathematical perspective on the phenomenology of non-perturbative Quantum Field Theory" by Ali Shojaei-Fard is also an ODE for a Lagrangian.

Fermat's principle of least action (actually, stationary rather than least) applied to light with the velocity in a medium, $v(x)$, inversely proportional to the refractive index, $n(x)$ (specifically, $v(x) = c/n(x)$, with c the speed of light in a vacuum) asserts that the path $x(t)$ of a light ray minimizes (or extremizes) the time of traversal from an initial point x_i to a final point x_f determined by

$$\int_{x_i}^{x_f} \frac{dx}{\pm\sqrt{\frac{2}{m}\ (E-U(x))}} = \int_{x_i}^{x_f} \frac{dx}{v(x)}$$

$$= \int_{x_i(t_i)}^{x_f(t_f)} \frac{dx}{\dot{x}(t)} = t_f - t_i = \Delta t.$$

Appendix 7: Reflections on analytic and geometric dualities of inverse pairs

To develop some facility with the notation and concepts, let's look at the interplay between the algebraic and differential analysis and the geometry of local inverse pairs of functions in some simple cases.

In this appendix, the notation $y(x)$ indicates a functional dependence on the value of x whereas y without an argument indicates no functional dependence on any variable. The subscripts of x_p and y_p indicate that these are the x and y coordinates of a point on the curve $y(x) = h(x)$.

Define two functions

$y(x) = h(x)$ and $y(x) = h^{(-1)}(x)$

to be a local inverse pair if, in a neighborhood about the point

$$(x_p, y_p) = (x_p, h(x_p)) = (h^{(-1)}(y_p), y_p) = (h^{(-1)}(y_p), h(x_p)),$$

the relation

$$h(h^{(-1)}(y)) = y \leftrightarrow x = h^{(-1)}(h(x))$$

holds. This may be written more concisely for a neighborhood about the origin for which $h(0) = 0$ and $h'(0) \neq 0$ or more generally for $h(0)$ not a local extremum as $h(h^{(-1)}(u)) = u = h^{(-1)}(h(u)).$

Since the inversion curve is defined by the interchange of the abscissas and ordinates (the coordinates) of the points of the functional curve $y(x) = h(x)$, the two curves and, therefore, their tangents are reflections of each other through the bisecting line $y(x) = x.$ This reflection property holds globally for any function $h(x)$, so the associated local inverse functions are sections of this reflected curve on each of which bijectivity (one-to-one correspondence of the abscissa and ordinate) is satisfied. The singular points are local extrema of the function $h(x)$, marking the boundaries of the ranges of the local inverse functions, and for a local finite extremum the derivative of $h(x)$ vanishes (back to finding zero). (We could argue geometrically that where $h(x)$ tends to infinity, the derivative vanishes as well with an extremum where the function tends to infinity through either positive or negative values on both sides of the singularity and an inflection point when it tends to infinity with opposite signs on opposing sides of the singularity.) Note inflection points, at which $h'(x)$ also vanishes, do not mark boundaries for the local inverses.

Differentiation of the conjugate inversion equations gives

$$1 = h'(h^{(-1)}(y))\ (h^{(-1)})'(y) = (h^{(-1)})'(h(x))\ h'(x),$$

implying the tangent vector to the curve $y = h(x)$ at the point $(x_p, y_p) = (x_p, h(x_p))$ is

$$(1,\ h'(x_p)) = (1,\ \frac{1}{(h^{(-1)})'(h(x_p))}) = (1,\ \frac{1}{(h^{(-1)})'(y_p)})$$

and the vector orthogonal to this curve is

$$(1,\ -\frac{1}{h'(x_p))}) = (1,\ -\frac{1}{h'(h^{(-1)}(y_p))}) = (1,\ -(h^{(-1)})'(y_p)),$$

which is the tangent to the curve $y(x) = -h^{(-1)}(x)$ at the point $(y_p, -h^{(-1)}(y_p)) = (h(x_p), -h^{(-1)}(h(x_p)))$. This curve, whose tangents are normal to the corresponding section of the curve $y(x) = h(x)$, is obtained by two successive reflections--the first through $y(x) = x$, the second, $y(x) = 0$.

Analytically reprising: for the curve $y(x) = h(x)$, the point (x_p, y_p) on the curve, the local inverse $y(x) = h^{(-1)}(x)$ and the reciprocals of their derivatives $g(x) = \frac{1}{f'(x)}$ and $\bar{g}(x) = \frac{1}{(g^{(-1)})'(x)}$,

the reciprocals of the derivatives are related by

$$h'(x_p) = \frac{1}{(h^{(-1)})'(h(x_p)}$$

$$= h'(h^{(-1)}(y_p) = \frac{1}{(h^{(-1)})'(y_p)}$$

$$= \frac{1}{g(x_p)} = \bar{g}(y_p),$$

the tangent vector at the point $(x_p,\ y_p) = (x_p,\ h(x_p))$ on the curve is

$$(1,\ h'(x_p)) = (1,\ \frac{1}{(h^{(-1)})'(h(x_p))}) = (1,\ \frac{1}{(h^{(-1)})'(y_p)})$$

$$= (1,\ \frac{1}{g(x_p)}) = (1,\ \bar{g}(y_p)).$$

and the corresponding normal is

$$(1,\ -\frac{1}{h'(x_p))}) = (1,\ -(h^{(-1)})'(h(x_p))) = (1,\ -(h^{(-1)})'(y_p))$$

$$= (1,\ -g(x_p)) = (1,\ -\frac{1}{\bar{g}(y_p)})$$

while

for the curve $y(x) = h^{(-1)}(x)$,

the tangent vector at the point $(y_p,\ x_p) = (h(x_p),\ x_p) = (y_p,\ h^{(-1)}(y_p))$ on the curve is

$$(1,\ (h^{(-1)})'(y_p)) = (1,\ (h^{(-1)})'(h(x_p))) = (1,\ \frac{1}{h'(x_p)})$$

$$(1,\ \frac{1}{\bar{g}(y_p)}) = (1,\ g(x_p)),$$

and the normal is

$$(1,\ -\frac{1}{(h^{(-1)})'(y_p)}) = (1,\ -\frac{1}{(h^{(-1)})'(h(x_p)))} = (1,\ -h'(x_p))$$

$$= (1,\ -\bar{g}(y_p))) = (1,\ -\frac{1}{g(x_p)}).$$

The normality condition is expressed analytically as the vanishing of the inner product

$$0 = (1,\ h'(x_p)) \cdot (1,\ -\frac{1}{h'(x_p)})$$

$$= (1,\ h'(x_p)) \cdot (1,\ -(h^{(-1)})'(y_p))$$

$$= (1,\ h'(x_p)) \cdot (1,\ -(h^{(-1)})'(h(x_p)))$$

$$= (1,\ \frac{1}{(h^{(-1)})'(y_p))}) \cdot (1,\ -(h^{(-1)})'(y_p))$$

$$= (1,\ \frac{1}{(h^{(-1)})'(h(x_p))}) \cdot (1,\ -(h^{(-1)})'(h(x_p)))$$

$$= (1,\ \frac{1}{(h^{(-1)})'(y_p))}) \cdot (1,\ -\frac{1}{h'(x_p)}).$$

The unlinked, nontrivial normality conditions true for any x or y in the local bijective sections are

$$(1,\ h'(x)) \cdot (1,\ -(h^{(-1)})'(h(x))) = 0,$$

$$(1,\ h'(h_+^{(-1)}(y)) \cdot (1,\ -(h^{(-1)})'(y)) = 0,$$

$$(1, g(x)) \cdot (1,\ -\bar{g}(h(x))) = 0,$$

$$(1, g(h^{(-1)}(y))) \cdot (1,\ -\bar{g}(y)) = 0.$$

Keep in mind that if the normality equation mixes x and y, they can not be independently chosen even within the bijective section but rather must be related as the coordinates of a point on the curve $y(x) = h(x)$.

Specific examples:

A) $y(x) = h(x) = x$ and $x(y) = h(y) = y$ are a global inverse pair.

The derivatives are

$h'(x) = 1$ and $h'(y) = 1$.

The derivative orthogonality conditions are

$$h'(h^{(-1)}(y))\ (h^{(-1)})'(y) = 1 \cdot 1 = 1$$

and

$$(h^{(-1)})'(h(x))\ h'(x) = 1 \cdot 1 = 1.$$

The vector orthogonality condition is the vanishing inner product of tangent vectors

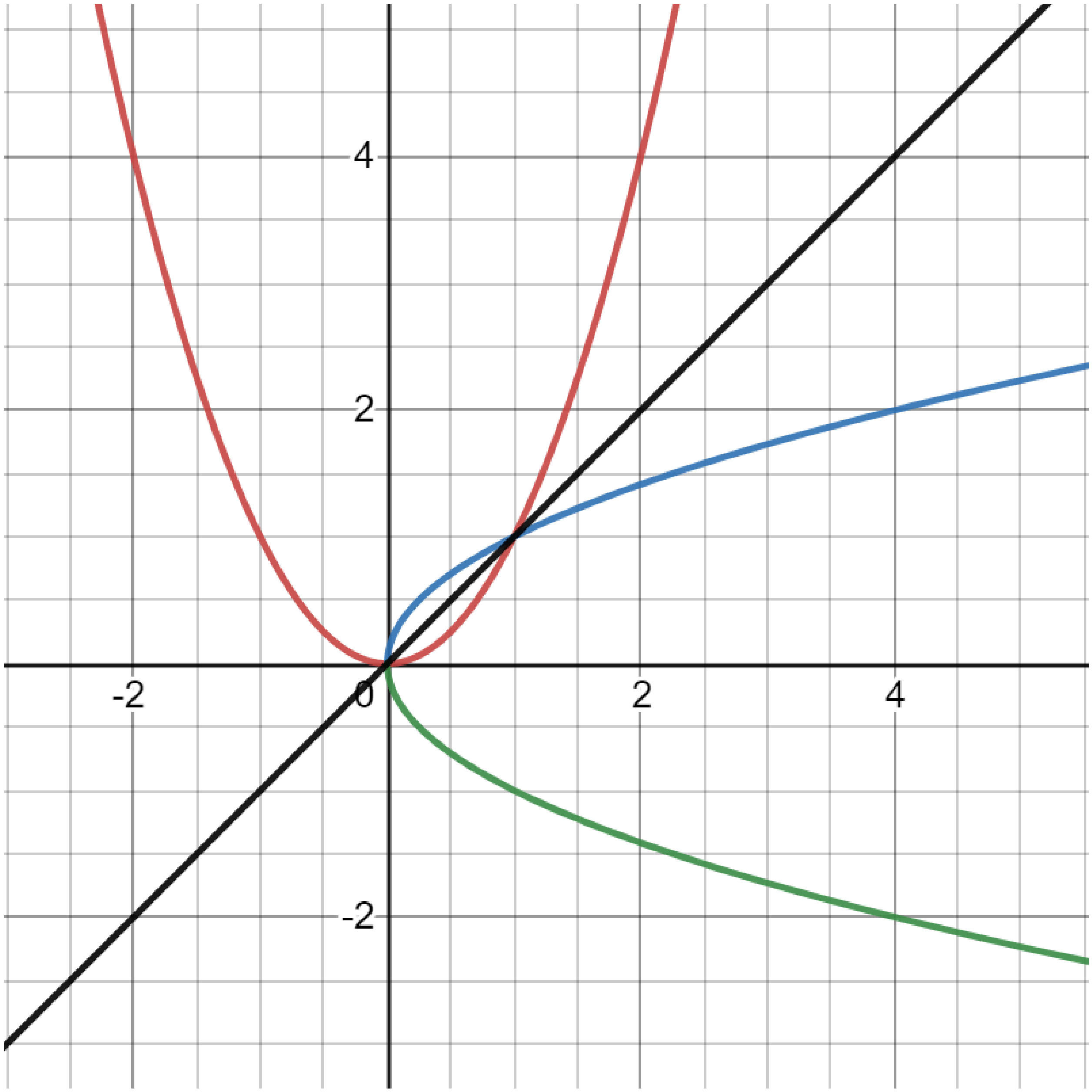
$$(1,\ h'(u)) \cdot (1,\ -(h^{(-1)})'(u)) = (1,\ 1) \cdot (1,\ -1) = 0.$$

B) The inverse curve to the curve / mapping

$y(x) = h(x) = x^2$, a function with the reals as the domain, is the curve,

$(y(x))^2 = x$, which is not a function and has the restricted domain $x \geq 0$.

A Desmos plot about the origin of the curve $y(x) = x^2$ and its reflection through $y(x) = x$, i.e., $y^2 = x(y)$, gives:



The red curve is $y = f(x) = x^2$. The blue curve is $y = \sqrt{x}$, the reflection of the red curve in the first quadrant through the black diagonal line $y = x$, The green curve is $y = -\sqrt{x}$, the reflection of the red curve in the second quadrant through the black diagonal line. The combined green and blue curves form the local inverse patchwork for $f(x) = x^2$; more precisely, given the point $(x_p, y_p) = (x_p, x_p^2)$ on $y = f(x) = x^2$, the reflected, or inverse, point is $(x_i, y_i) = (y_p, x_p) = (y_p, sign(x_p)\sqrt{y_p})$.

In other words, for $x_p \neq 0$, i.e., omitting the minimum of the curve $y(x) = h(x) = x^2$, the two functional branches of the inverse curve are determined by the sign of x_p of the point $(x_p,\ y_p)$ on the curve $y(x) = h(x) = x^2$; more specifically, the two functional inverse branches / sections are given by $y_{sign(x_p)}(x) = sign(x_p)\sqrt{sign(x_p)\ x}$.

Let's examine the two bijective sections.

The inverse curve to the curve

$y(x) = h(x) = x^2$ in the first quadrant ($x, y > 0$ of the xy-plane) is the curve

$y(x) = h_+^{(-1)}(x) = \sqrt{x}.$ also in the first quadrant.

Then the inverse function pair for $x, y > 0$ is

$y(x) = h(x) = x^2$ and

$y(x) = h_+^{(-1)}(x) = \sqrt{x}.$

The derivatives are

$h'(x) = 2x$ and

$(h_+^{(-1)})'(x) = \frac{1}{2\sqrt{x}}$.

The derivative reciprocal conditions are

$h'(h_+^{(-1)}(x)) \cdot (h_+^{(-1)})'(x) = 2\ (\sqrt{x}) \cdot \frac{1}{2\sqrt{x}} = 1$

and

$(h_+^{(-1)})'(h(x)) \cdot h'(x) = \frac{1}{2\sqrt{x^2}} \cdot 2x = 1$.

The two equivalent vector orthogonality conditions are

$(1,\ h'(x)) \cdot (1,\ -(h_+^{(-1)})'(h(x))) = (1,\ 2x) \cdot (1,\ \frac{1}{-2\sqrt{x^2}}) = 0$

and

$$(1,\ \frac{1}{(h_+^{(-1)})'(x))}) \cdot (1,\ -\frac{1}{h'(h_+^{(-1)}(x))}) = (1,\ 2\ \sqrt{x}) \cdot (1,\ -\frac{1}{2\ \sqrt{x}}) = 0.$$

The inverse curve to the curve

$y(x) = x^2$ in the second quadrant ($x < 0$ and $y > 0$) is

$y(x) = -\sqrt{x}$ in the fourth quadrant ($x > 0$ and $y < 0$).

The inverse function pair is

$y(x) = h(x) = x^2$ for $x < 0$ and

$y(x) = h_-^{(-1)}(x) = -\sqrt{x}$ for $x > 0$.

Check:

$h(h_-^{(-1)}(x)) = (-\sqrt{x})^2 = x$ for $x > 0$ and

$h_-^{(-1)}(h(x)) = -\sqrt{x^2} = -|x| = x$ for $x < 0$.

The derivatives are

$h'(x) = 2x$ for $x < 0$ and

$(h_-^{(-1)})'(x) = -\frac{1}{2\sqrt{x}}$ for $x > 0$.

The derivative reciprocal conditions are

$h'(h_-^{(-1)}(x)) \cdot (h_-^{(-1)})'(x) = 2\ (-\sqrt{x}) \cdot (-\frac{1}{2\sqrt{x}}) = 1$ for $x > 0$ and

$(h_-^{(-1)})'(h(x)) \cdot h'(x) = -\frac{1}{2\sqrt{x^2}} \cdot 2x$

$= -\frac{1}{2|x|} \cdot 2x = \frac{1}{2x} \cdot 2x = 1$ for $x < 0$.

The vector orthogonality conditions are, for $x < 0$,

$(1,\ h'(x)) \cdot (1,\ -(h_{-}^{(-1)})'(h(x))) = (1,\ 2x) \cdot (1,\ \frac{1}{2\sqrt{x^2}}) = 1 + \frac{x}{|x|}$

$= 1 + sign(x) = 1 - 1 = 0$

and, for $x > 0$,

$(1,\ \frac{1}{(h_{-}^{(-1)})'(x))}) \cdot (1,\ -\frac{1}{h'(h_{-}^{(-1)}(x))})$

$= (1,\ -2\sqrt{x}) \cdot (1,\ \frac{1}{2\sqrt{x}}) = 0.$

C. The curve inverse to the curve

$y(x) = \sin(\frac{\pi}{2}x)$, a function for all real x is

$x(y) = \sin(\frac{\pi}{2}x)$ for $|x| \leq 1$, not a function.

The inverse curve to the curve

$y(x) = h(x) = \sin(\frac{\pi}{2}x)$ for $-1 < x < 1$ is

$y(x) = h_0^{(-1)}(x) = \frac{2}{\pi}\ \arcsin(x)$ for $-1 < x < 1$.

The derivatives are

$h'(x) = \frac{\pi}{2}\ \cos(\frac{\pi}{2}x)$ and

$(h_0^{(-1)})'(x) = \frac{2}{\pi}\ \frac{1}{\sqrt{1-x^2}}$.

The reciprocal relations of the derivatives are

$$h'(h_0^{(-1)}(x)) \cdot (h_0^{(-1)})'(x) = \frac{\pi}{2}\ \cos(\arcsin(x)) \cdot \frac{2}{\pi}\ \frac{1}{\sqrt{1-x^2}} = 1$$

and

$$(h_0^{(-1)})'(h(x)) \cdot h'(x) = \frac{2}{\pi}\ \frac{1}{\sqrt{1-(\sin(\frac{\pi}{2}x))^2}} \cdot \frac{\pi}{2}\ \cos(\frac{\pi}{2}x) = 1.$$

The validity of the second reciprocal relation is obvious, and the first follows from substitution in the second of $h^{(-1)}(x)$ for x (corroborated numerically by a plot).

The vector orthogonality conditions for $|x| < 1$ are, of course, satisfied since the reciprocal relations hold, so I will not illustrate these again.

The inverse pair for an increase of one period is

$y(x) = h(x) = \sin(\frac{\pi}{2}x)$ for $1 < x < 3$ and

$y(x) = h_1^{(-1)}(x) + 2 = -\frac{2}{\pi}\ \arcsin(x) + 2$ for $-1 < x < 1$.

The inverse pair for an decrease of one period is

$y(x) = h(x) = \sin(\frac{\pi}{2}x)$ for $-3 < x < -1$ and

$y(x) = h_{-1}^{(-1)}(x) = -\frac{2}{\pi}\ \arcsin(x) - 2$ for $-1 < x < 1$.

The general inverse pairs for $n = \ldots,\ -2,\ -1,\ 0,\ 1,\ 2,\ \ldots$ are

$y(x) = h(x) = \sin(\frac{\pi}{2}x)$ for $-1 + 2n < x < 1 + 2n$ and

$y(x) = h_n^{(-1)}(x) = (-1)^n\ \frac{2}{\pi}\ \arcsin(x) + 2n$ for $-1 < x < 1$. ?????? re-check ?????

For these inverse pairs,

for all x,

$g(x) = \frac{1}{h'(x)} = \frac{1}{\frac{\pi}{2}\ \cos(\frac{\pi}{2}x)}$

and, for $-1 < x < 1$,

$\bar{g}_n(x) = \frac{1}{(h_n^{(-1)})'(x)} = (-1)^n \frac{\pi}{2}\sqrt{1 - x^2}$.

Now overlay some fundamental physics on the math.

Return to

$y(x) = h(x) = \sin(\frac{\pi}{2}x)$ for $-1 < x < 1$ and

$y(x) = h_0^{(-1)}(x) = \frac{2}{\pi}\ \arcsin(x)$ for $-1 < x < 1$

and identify

$h(u) = \sin(\frac{\pi}{2}u)$ for $-1 < u < 1$

with

$\frac{x(t)}{x_m} = \sin(\frac{\pi}{2}\frac{t}{t_m})$ for $-1 < \frac{t}{t_m} < 1$

as the trajectory of a mass m attached to an ideal spring with spring constant k, a harmonic oscillator, with maximum displacement x_m and $u = \frac{v_m}{x_m}t = \frac{t}{t_m}$, where $v_m > 0$ is the maximum velocity of the mass at time $t = 0$, and t_m is the quarter period of oscillation. Then

$\frac{t(x)}{t_m} = \frac{2}{\pi}\ \arcsin(\frac{x}{x_m})$ for $-1 < \frac{x}{x_m} < 1$.

Introduce a link to Newtonian, Hamiltonian, Lagrangian, and Hamilton-Jacobi formulations of physics by identifying geometry with physics via

$$\bar{g}(x) = \frac{1}{\partial_{\frac{x}{x_m}}(\frac{t(x)}{t_m})} = \frac{2}{\pi}\ t_m\ \sqrt{1 - (\frac{x}{x_m})^2}$$

by noting that the total energy $E = T + U$ of the mass remains constant while its potential energy $U = kx^2$ (also often denoted as V) and kinetic energy $T = \frac{v^2}{2m}$ fluctuate. The total energy is determined by the maximum speed of the mass at the origin v_m, where the potential is zero, as $E = \frac{v_m^2}{2m}$ or as its potential energy at maximum displacement $E = kx_m^2$, when the velocity and, therefore, kinetic energy is zero. The kinetic energy of the mass then is $T = \frac{mv^2}{2}\ = E - U$, implying in our case with $v_m > 0$ at time $t = 0$ that during the half period of motion $-1 < \frac{t}{t_m} < 1$ between turning points the velocity as a function of position of the mass is

$$\frac{dx(t)}{dt} = \dot{x}(t) = v(x) = \sqrt{\frac{2T}{m}} = \frac{2}{m}\sqrt{E-U}$$

$$= \sqrt{\frac{2E}{m}}\ \sqrt{1-\frac{U}{E}} = \sqrt{\frac{2T_m}{m}}\ \sqrt{1-\frac{U}{E}}$$

$$= v_m\ \sqrt{1-\frac{U}{E}} = v_m\ \sqrt{1-\frac{kx^2}{kx_m^2}} = v_m\ \sqrt{1-(\frac{x}{x_m})^2}.$$

This gives us our physical autonomous ODE for the flow vector, i.e., the velocity

$$\dot{x}(t) = v(x) = v_m\ \sqrt{1-(\frac{x}{x_m})^2},$$

which is a physical realization of our mathematical formalism
'

$$g(x) = \frac{1}{\partial_x\ h(x)} = \partial_y h^{(-1)}(y) = \partial_y x(y) = g(x)$$

with the allied geometrics.

For other potential energy fields in which the total energy of the particle is conserved, i.e., the energy is independent of time, and the potential energy depends only on position, we can apply the same mathematical formalism if we break down the trajectory into segments where bijectivity between the velocity and position holds, i.e., between turning points, where the velocity is zero (inflection points are no problem). Below I show how these observations are related to stationary principles.

A Legendre transform also applies to compositional inverse pairs $y = h(x) \geq 0$ and $x = h^{(-1)}(y) \geq 0$ for $h(0) = 0$. A graphical representation of the area of the rectangle with lower left corner at the origin and upper right corner at $(x_p,\ y_p) = (x_p,\ h(x_p)) = (h^{(-1)}(y_p),\ y_p) = (h^{(-1)}(y_p),\ h(x_p))$, i.e., a point on the curve $y(x) = h(x) \geq 0$ gives the sum of the area between the curve and the x-axis and between the curve and the y-axis as

$$Area_x(x_p) + Area_y(y_p) = \int_0^{x_p}\ h(x)\ dx + \int_0^{y_p}\ h^{(-1)}(y)\ dy = x_p\ y_p,$$

which satisfies

$\partial_{x_p} \int_0^{x_p} h(x)\, dx = h(x_p) = y_p$ and

$\partial_{y_p} \int_0^{y_p} h^{(-1)}(y)\, dy = h^{(-1)}(y_p) = x_p$. This follows from the graphical representation.

Consequently, the Lengendre transform involutions

$$Area_x(x_p) = x_p\, y_p - Area_y(y_p)$$

and

$$Area_y(y_p) = x_p\, y_p - Area_x(x_p),$$

or

$$\int_0^{x_p} h(x)\, dx = x_p\, y_p - \int_0^{y_p} h^{(-1)}(y)\, dy$$

and

$$\int_0^{y_p} h^{(-1)}(y)\, dy = x_p\, y_p - \int_0^{x_p} h(x)\, dx$$

hold.

With the obvious meaning of notation, this may be re-expressed as

$z(y) = \bar{f}(y) = xy - f(x)$ with $x = h^{(-1)}(y) = \frac{\partial \bar{f}(y)}{\partial y}$.

The tangent line to the curve $z(y) = \bar{f}(y)$ at the point $(y_p, z_p) = (y_p, \bar{f}(y))$ has z-intercept $-f(x)$ and slope x, so knowing only $f(x)$ allows us to ensconce the unknown curve $z(y) = \bar{f}(y)$ in an envelope of tangent lines.

For example, for the inverse pair

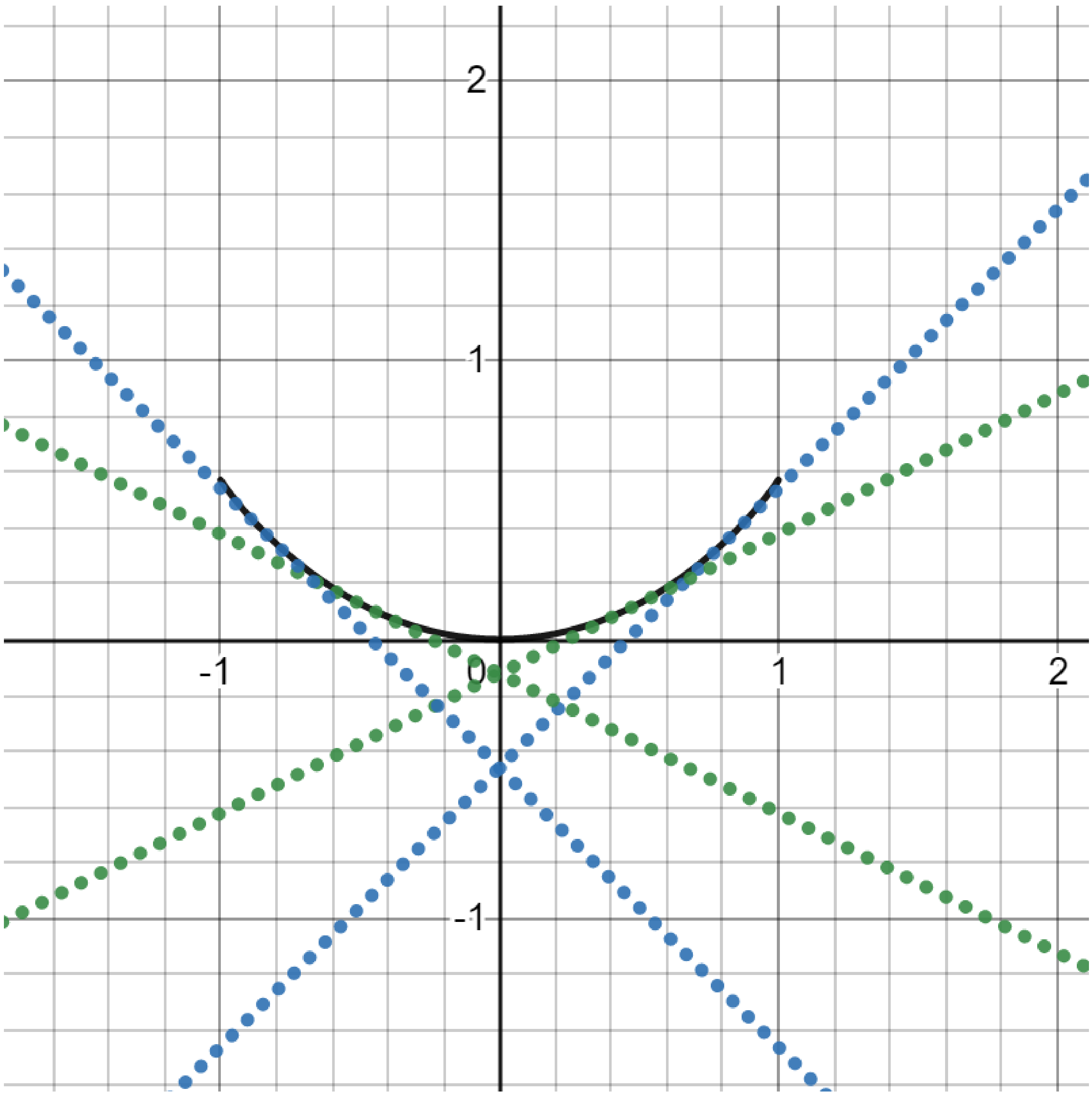
$y(x) = h(x) = \sin(x)$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$ and

$x(y) = h^{(-1)}(y) = \arcsin(y)$ for $1 < x < -1$, the associated integrals are

$f(x) = -\cos(x) + 1$ and

$\bar{f}(y) = \sqrt{1-y^2} + y\arcsin(y) - 1$.

Then the tangent lines, parametrized by x, to the curve $z(y) = \bar{f}(y)$ are $z(y) = xy - f(x) = xy + \cos(x) - 1$ for $-\frac{\pi}{2} < x < \frac{\pi}{2}$, and we have the envelope of four tangent lines in the following Desmos plot;



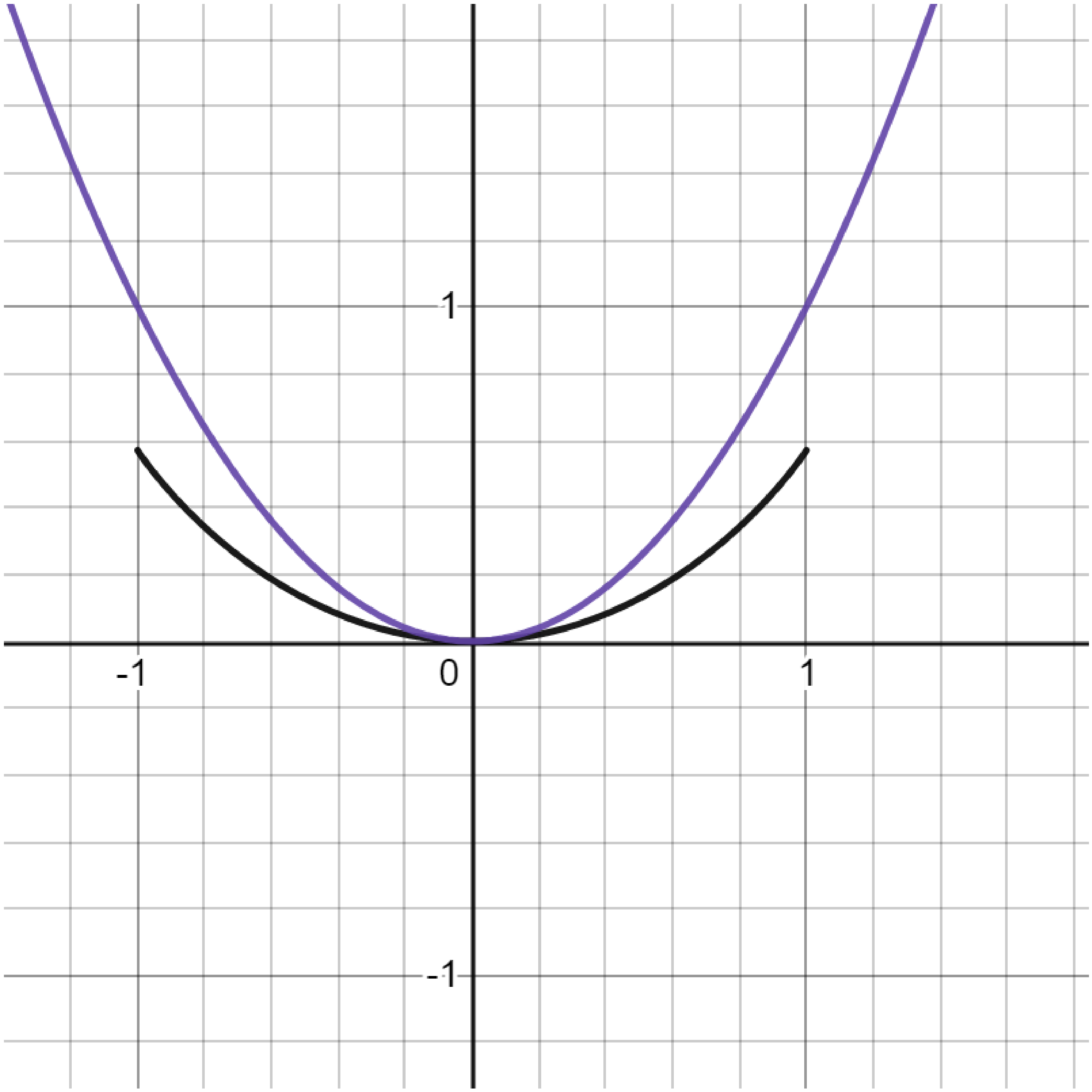
with

$z = \bar{f}(y) = \sqrt{1-y^2} + x \ \arcsin() - 1$ is the black curve from $-1 \leq y \leq 1$,

$z = \pm y + \cos(\pm 1) - 1$ are the blue dotted tangent lines with $x = \pm 1$, and

$z = \pm .5y + \cos(\pm .5) - 1$ are the green dotted tangent lines with $x = \pm .5$.

The curvature of $z(y) = \bar{f}(y)$ is $z''(y) = (\bar{h}^{(-1)})'(y) = \frac{1}{h'(x)}$ where $y = h(x)$, so we could approximate $z(y) = \bar{f}(y)$ around the origin with $z(y) = \frac{1}{h'(0)}\ y^2$, which in the example is $z(y) = \frac{1}{cos(x)}|_{x=0}\ y^2 = y^2$, depicted in the following plot as the blue curve (the black is $\bar{f}(y)$:



Of course, we could use the iterated infinigen to get better and better approximations (truncated Taylor series) of $\bar{f}(y)$.

If we add a quantity independent of x_p and y_p, say $U(q)$, to both sides of any of the Legendre transform equalities, the equalities remain valid, for example,

$$\int_0^{y_p} h^{(-1)}(y)\, dy + U(q) = x_p\, y_p - (\int_0^{x_p} h(x)\, dx - U(q));$$

in particular, this holds when

$h(p) = \frac{\partial H}{\partial p} = \frac{\partial T}{\partial p} = \frac{p}{m} = v$ and
$h^{(-1)}(v) = \frac{\partial L}{\partial v} = \frac{\partial T}{\partial v} = mv = p$,

where H and L are the Hamiltonian and the Lagrangian and the potential $U(q)$ is independent of p and v. This satisfies the general reciprocal derivative relation for an inverse pair as $\partial_p \; h(p) = \frac{\partial^2 H}{\partial p^2} = \frac{1}{m}$ and $\partial_v \; h^{(-1)}(v) = \frac{\partial^2 L}{\partial v^2} = m$.

For a free particle moving in one dimension with constant momentum p and, therefore, constant total energy E, the non-relativistic quantum probability amplitude is

$A(x,t;E,p) = e^{i(px-Et)/\hbar}$,

where the phase is regulated by Planck's constant $\hbar$. The particle has equal relative probability of being detected anywhere at any time. The phase can be identified as an action and Feynman path integration as integration of the probability amplitude over different paths over which the action can vary. See . . . .

Appendix 8: Characterizing the flow function without using the infingen

In this section, I delineate the basic properties of the flow function without resorting to an iterated infinigen. This veils the connection to Lie theory and the infinigen series of equations (B.21) and (B.26) of Gell-Mann and Low demonstrated in the following and other sections.

With the flow function defined as

$$F(t,z) = f^{(-1)}[t + f(z)] \; ,$$

evaluating at different values gives

$$F(0,z) = f^{(-1)}[f(z)] = z$$

and

$$F(t,0) = f^{(-1)}[t + f(0)],$$

and taking derivatives gives

$$\frac{\partial F(t,z)}{\partial t} = (f^{(-1)})'[t + f(z)]$$

and

$$\frac{\partial F(t,z)}{\partial z} = (f^{(-1)})'[t + f(z)]\ f'(z).$$

The derivatives imply the tangency condition

$$\frac{\partial F(t,z)}{\partial t} - \frac{1}{f'(x)}\ \frac{\partial F(t,z)}{\partial z}$$

$$= \frac{\partial F(t,z)}{\partial t} - g(z)\ \frac{\partial F(t,z)}{\partial z}$$

$$= (f^{(-1)})'[t + f(z)] - (f^{(-1)})'[t + f(z)] = 0.$$

with

$$g(x) = \frac{1}{f'(x)}.$$

Differentiation of the inversion relation gives

$$\frac{\partial F(0,z)}{\partial z} = \frac{\partial}{\partial z}\ f^{(-1)}[f(z)] = (f^{(-1)})'[f(z)]\ f'(z) = \frac{\partial}{\partial z}\ z = 1,$$

i.e., the reciprocal relation of the derivatives of the inverse pair

$$(f^{(-1)})'[f(z)] = \frac{1}{f'(z)} = g(z).$$

Substitution gives the autonomous O.D.E.

$$(f^{(-1)})'(z) = \frac{1}{f'(f^{(-1)}(z))} = g(f^{(-1)}(z)).$$

From the derivative w.r.t. t evaluated at $t = 0$ and the reciprocal properties of the derivatives,

$$\frac{\partial F(t,z)}{\partial t} \,|_{t=0} = (f^{(-1)})'[f(z)] = \frac{1}{f'(z)} = g(z).$$

Integration gives

$$f(z) + C = \int f'(z)\, dz = \int \frac{1}{g(z)}\, dz.$$

Alternatively, from the tangency property,

$$0 = [\frac{\partial F(t,z)}{\partial t} - g(z)\, \frac{\partial F(t,z)}{\partial z}] \,|_{t=0}$$

$$= \frac{\partial F(t,z)}{\partial t} \,|_{t=0} - g(z)\, \frac{\partial z}{\partial z}$$

$$= (f^{(-1)})'(f(z)) - g(z).$$

When well-defined, the translation group property

$$F(t, F(s,z)) = F(t, f^{(-1)}[s + f(z)]) = f^{(-1)}[t + f(f^{(-1)}[t + f(z)])]$$

$$= f^{(-1)}[t + s + f(z)] = F(t+s, z)$$

holds.

Kuczma on pg. 20 in section “10. The equation of translation” in “A survey of the theory of functional equations” gives $F(t,x) = f^{(-1)}[t + f(z)]$, where $f(x)$ ia an arbitrary continuous and strictly increasing function, as the general continuous solution of the functional equation $F(t, F(s,z)) = F(t+x, z)$. (See also eqn. (5) on pg. 24 of “The work of Niels Henrik Abel” by Houzel.) In section “17. Iteration” Kuczma discusses identities of the form $f^u(x) = \phi^{(-1)}[u + \phi(x)]$ in relation to compositional iteration and the Abel and Schröder equations. On pg. 6, he gives the solution $\phi(x,y) = f^{(-1)}[f(x)+f(y)]$ (cf. formal group laws) as the solution to the functional equation $\phi(x,\phi(y,z)) = \phi(\phi(x,y),z)$ and $\phi(x,y) = f^{(-1)}[g(x)+f(y)]$ for $\phi(x,\phi(y,z)) = \phi(y,\phi(x,z))$ for arbitrary functions $f(x)$ and $g(x)$.

For a presentation of the flow function as an extremal of a functional, see pgs. 260 through 263 of "Variational aspects of the Abel and Schröder functional equations" by McKiernan.

See also "The continuous iteration of real functions" by Ward and Fuller.

Appendix 9: Characterizing the flow function via the iterated infinigen

Given the function $f(x)$ and its local inverse $f^{(-1)}(x)$, define the infinigen

$$g(x)\,\frac{\partial}{\partial x} = \frac{1}{f'(x)}\,\frac{\partial}{\partial x} = \frac{\partial}{\partial f(x)} = \frac{\partial}{\partial y}.$$

Then given an analytic function $W(x)$ at x, for a suitable range of t,

$$\exp[t\;g(x)\,\frac{\partial}{\partial x}]\;W(x) = \exp[t\;\frac{\partial}{\partial y}\;]\;W(f^{(-1)}(y)) = W(f^{(-1)}(t+f(x)))$$

$$= W[F(t,x)]$$

with

$$F(t,x) = f^{(-1)}(t+f(x)).$$

This function $\overline{W}(t,x) = W(F(t,x))$ inherits only a limited translation group property from that of F; specifically,

$$\overline{W}(t,F(s,x)) = W[F(t,F(s,x))] = W[F(t+s,x)] = \overline{W}(t+s,x),$$

and, in general,

$$\overline{W}(t,\overline{W}(s,x)) \neq \overline{W}(t+s,x).$$

The full tangency property

$$\frac{\partial}{\partial t}\overline{W}(t,x) - g(x)\ \frac{\partial}{\partial x}\overline{W}(t,x) = 0$$

does follow from differentiation of the exponentiated infinigen as it does for F.

For $W(x) = x$, analytic continuation gives over sections of bijectivity of the inverse pair

$$\exp[t\ g(x)\ \frac{\partial}{\partial x}]\ x = f^{(-1)}(t + f(x)) = F(t,x).$$

Then, for $y(x) = f(x)$, repeated differentiation w.r.t. t gives, at $t = 0$,

$$[g(x)\ \frac{\partial}{\partial x}]^n\ x = (f^{(-1)})^{(n)}(f(x)) = \frac{\partial^n F(t,x)}{\partial t^n}\ |_{t=0}$$

$$= \frac{\partial^n}{\partial y^n}\ f^{(-1)}(y) = (f^{(-1)})^{(n)}(y) = (f^{(-1)})^{(n)}(f(x)),$$

and, in particular, for $n = 0$,

$$x = F(0,x)$$

and, for $n = 1$,

$$\frac{1}{f'(x)} = g(x) = (f^{(-1)})'(f(x)) = \frac{\partial F(t,x)}{\partial t}\ |_{t=0} = (f^{(-1)})'(y),$$

giving the reciprocal property of the first derivatives of the local inverse pair, i.e., the inverse function theorem.

Substitution in this last equation gives the autonomous O.D.E.

$$\frac{1}{f'(f^{(-1)}(x))} = g(f^{(-1)}(x)) = (f^{(-1)})'(x)$$

$$= \frac{\partial F(t, f^{(-1)}(x))}{\partial t}\ |_{t=0} = \frac{\partial f^{(-1)}(t+x))}{\partial t}\ |_{t=0} = (f^{(-1)})'(x).$$

Integration gives

$$f(x) + constant = \int f'(x)\, dx = \int \frac{1}{g(x)}\, dx.$$

The tangency property follows from the first derivative as well;

$$\frac{\partial F(t,x)}{\partial t} = \frac{\partial}{\partial t} \exp[t\ g(x) \frac{\partial}{\partial x}]\ x$$

$$= g(x) \frac{\partial}{\partial x} \exp[t\ g(x) \frac{\partial}{\partial x}]\ x = g(x) \frac{\partial F(t,x)}{\partial x},$$

giving the tangency property

$$\frac{\partial F(t,x)}{\partial t} - g(x) \frac{\partial F(t,x)}{\partial x} = 0.$$

Alternatively, the derivative w.r.t. x gives the tangency property as

$$\frac{\partial F(t,x)}{\partial x} = \frac{\partial}{\partial x} \exp[t\ g(x) \frac{\partial}{\partial x}]\ x$$

$$= \frac{1}{g(x)} g(x) \frac{\partial}{\partial x} \exp[t\ g(x) \frac{\partial}{\partial x}]\ x$$

$$= \frac{1}{g(x)} \frac{\partial}{\partial t} \exp[t\ g(x) \frac{\partial}{\partial x}]\ x = \frac{1}{g(x)} \frac{\partial F(t,x)}{\partial t}.$$

The translation equation follows from

$$\exp[s\ g(x) \frac{\partial}{\partial x}]\ \exp[t\ g(x) \frac{\partial}{\partial x}]\ x$$

$$= \exp[t\ g(x) \frac{\partial}{\partial x}]\ F(t,x) = F(t, F(s,x))$$

$$= \exp[(t+s)\ g(x) \frac{\partial}{\partial x}]\ x = F(t+s, x).$$

In Gell-Mann and Low (as suggested by T. D. Lee), the infinigen series appears in the equations (B.21) and (B.26) on pg. 3112, and in K & M, the equation (54) on pg. 14 for vector functions.

See the refs in OEIS A145271 and the associated A133437, A134264, A134685, A139002, and A139605 for the appearance of various manifestations of the infinigen series in diverse areas of mathematics and physics.

Appendix 10: Characterizing the flow function via the translation functional equation

Given sufficient continuity and the translation group property

$$F[t,\ F(s,z)] = F(t+s,\ z),$$

then

$$F[0,\ F(s,\ z)] = F(s,\ z)$$

and

$$F(0,z) = z.$$

Given sufficient analyticity, define

$$g(z) := \frac{\partial F(t,z)}{\partial t}\ |_{t=0}.$$

Then

$$g(F(s,z)) = \frac{\partial F[t,F(s,z)]}{\partial t}\ |_{t=0} = \frac{\partial F[t+s,z]}{\partial t}\ |_{t=0}$$

$$= \frac{\partial F[t+s,z]}{\partial s}\ |_{t=0} = \frac{\partial F[t,F(s,z)]}{\partial s}\ |_{t=0}$$

$$= \frac{\partial F[0,F(s,z)]}{\partial s} = \frac{\partial F(s,z)}{\partial s},$$

giving the diff eq

$$\frac{\partial F(t,z)}{\partial t} = g(F(t,z)).$$

The tangency condition may be derived from

$$\frac{\partial F[t+s,z]}{\partial s} = \frac{\partial F[t+s,z]}{\partial t} = \frac{\partial F[s,F(t,z)]}{\partial t} = \frac{\partial F[s,y]}{\partial y}\ \frac{\partial y}{\partial t}\ |_{y=F(t,z)}$$

by letting $t = 0$ and using $F(0, z) = z$, giving

$$\frac{\partial F[s,z]}{\partial s} = \frac{\partial F[s,z]}{\partial z} \; \frac{\partial F(t,z)}{\partial t} \; |_{t=0} = g(z) \; \frac{\partial F[s,z]}{\partial z},$$

so

$$\frac{\partial F[t,z]}{\partial t} - g(z) \; \frac{\partial F[t,z]}{\partial z} = 0.$$

Combining equations,

$$\frac{\partial F[t,z]}{\partial z} = \frac{\partial F[t,z]}{\partial t} / g(z) = \frac{g[F(t,z)]}{g(z)}.$$

These diff eqs are called the Azcel-Frege-Jabotinsky equations by some authors. See "Gottlob Frege, A Pioneer in Iteration" by Granau, "Some differential equations related to iteration theory" by Aczel and Granau, and "Eri Jabotinsky, mathematician and politician: a short biography" by Gronau. Frege also considered the infinitesimal generator approach. From the man himself, see "Analytic iteration" by Jabotinsky and "On analytic iteration" by Erdos and Jabotinsky.

Appendix 11: A flow function and the Ovsyannikov diff eq

In "Introduction to the Theory of Quantized Fields" (3rd ed.), the authors, Boboliubov and Shirkov, introduce the Ovsyannikov diff eq equation (eqn. (49.10) on pg. 505) for some charge-related quantities

$$[x \; \frac{\partial}{\partial x} + \varphi(h) \; \frac{\partial}{\partial h}] \; \bar{h}(x,h) = 0$$

with

$$\varphi(h) = \frac{\partial \bar{h}(x,h)}{\partial x} \; |_{x=1}$$

and

$$\bar{h}(1,h) = h.$$

They give the solution as the flow function

$$\bar{h}(x,h) = \Psi^{(-1)}[\ln(x) + \Psi(h)],$$

where

$$\Psi(h) = \int^h \frac{u}{\varphi(u)}.$$

To make contact with the infingen approach, consider

$$\exp[t\ x\ \frac{\partial}{\partial x}]\ \bar{h}(x,h) = \exp[-t\ \varphi(h)\ \frac{\partial}{\partial h})]\ \bar{h}(x,h),$$

where x and h are independent variables. With $y = \ln(x)$, $x\frac{\partial}{\partial x} = \frac{\partial}{\partial y}$ and our exponentiated ops, as in previous arguments, give

$$\bar{h}[e^t x, h] = \bar{h}[x, \Psi^{(-1)}(t + \Psi(h))],$$

a functional equation satisfied by

$$\bar{h}(x,h) = \Psi^{(-1)}[\ln(x) + \Psi(h)]$$

since

$$\bar{h}(e^t x, h) = \Psi^{(-1)}[t + \ln(x) + \Psi(h)]$$

and

$$\bar{h}[x, \Psi^{(-1)}(t + \Psi(h))] = \Psi^{(-1)}[\ln(x) + \Psi(\Psi^{(-1)}(t + \Psi(h)))]$$
$$= \Psi^{(-1)}[t + \ln(x) + \Psi(h)].$$

B & S call their eqn. (49.14)

$$\int_h^{\bar{h}(x,h)} \frac{d\alpha}{\varphi(\alpha)} = \ln(x)$$

the Gell-Mann--Low equation.This is also the integral (A.2) discussed by Shirkov on pg. 171 of Brown.

Appendix 12: Frege on functional iteration

This appendix contains some notes on Frege's habilitation on functional iteration as discussed in "Gottlob Frege, A Pioneer in Iteration" by Granau. It's a little confusing to distinguish when a quantity is to be regarded as a constant or as a variable in Frege's notation and the meaning of the symbols varies in the paper, but here is a mapping from the notation of Frege as given by Granau to a notation more amenable to comparison with that of this pdf;

$n \to \delta t,$

$f \to F,$

$\vartheta \to f,$

$X \to f^{(-1)}$

$\varphi \to g,$

$f(n_0, f(n_1, x)) = f(n_0 + n_1, x) \to F(t_0, F(t_1, x)) = F(t_0 + t_1, x),$

$\varphi(x) \to g(x) = \frac{1}{f'(x)},$

$f(n, x) = \vartheta^{-1}(n + \vartheta(x)) \to F(\delta t, x) = f^{(-1)}(\delta t + f(x)),$

$$n = \int \frac{dX}{\varphi(X)} = \vartheta(X) + C$$

$$\to \delta t(X, x) = \int_x^X \frac{du}{g(u)} = \int_x^{X(t)} f'(u)\, du = f(X(t)) - f(x),$$

$n = \vartheta(X) - \vartheta(x) \to \delta t(X, x) = f(X) - f(x),$

and, with x and t treated as independent variables,

$$\varphi(x) = \frac{\partial f(n, x)}{\partial n}\,|_{n=0} \to g(x) = \frac{\partial F(t, x)}{\partial t}\,|_{t=0},$$

$$\frac{\partial f(n, x)}{\partial n} = \varphi(f(n, x)) \to \frac{\partial F(t, x)}{\partial t} = g(F(t, x)),$$

$$\frac{\partial X}{\partial n} = \varphi(X) \to \frac{\partial X(t)}{\partial t} = g(X(t)) = \frac{\partial f^{-1}(t)}{\partial t} = g(f^{(-1)}(t)),$$

and

$$f(n) = x + \frac{\varphi_1}{1}\, n + \frac{\varphi_2}{1 \cdot 2}\, n^2 + \ldots \to f(\delta t) = \exp[\delta t\; g(x) \frac{\partial}{\partial x}]\; x \mid_{x=0}.$$

Referring to the equation above

$$n = \vartheta(X) - \vartheta(x) \to \delta t(X, x) = f(X) - f(x),$$

consistent with

$$F(\delta t, x) = f^{(-1)}(\delta t + f(x))$$

and

$$\delta t(X, x) = f(F(\delta t, x)) - f(x) = f(X) - f(x).$$

Relating these equations to integer $n \geq 0$ discrete iterations beginning with some initial value x, we have for $n = 2$, using the translation property,

$F(0, x) = f^{(-1)}(0 + f(x)) = x$,

$F(1, x) = F(1, F(0, x)) = f^{(-1)}(1 + f(x)),$

$F^{((2))}(1, x) := F(2, x) = F(1, F(1, x)) = F(1, F(1, F(0, x))) = f^{(-1)}(2 + f(x))$,

$\vdots$

$$F^{((n))}(1, x) := F(n, x) = F(1, F(1, F(1, ..., F(1, x))...) = f^{(-1)}(n + f(x))$$

with n iterations.

Generalizing,

$$X := F^{((\delta t))}(1, x) := f^{(-1)}(\delta t + f(x))$$

and

$$\delta t(X, x) = f(F^{((\delta t))}(1, x)) - f(x) = f(X) - f(x).$$

An example of an application of this methodology given by Frege and illustrated by Granau is for

$$\vartheta(x) = \arctan(\frac{x+b}{a}) \to f(x) = \arctan(\frac{x+b}{a}) = -\frac{i}{2}\ \ln(\frac{i-z}{i+z}),$$

$$n = \vartheta(X) - \vartheta(x) = \arctan)(\frac{X+b}{a}) - \arctan)(\frac{x+b}{a})$$

$$\to \delta t = f(X) - f(x) = \arctan(\frac{X+b}{a}) - \arctan(\frac{x+b}{a}),$$

with, in my notation, neglecting stipulating the branch of arctan for bijectivity for a true local inverse pair for given $a, b,$ and x,

$f(x) = \arctan(\frac{x+b}{a})$ and $f^{(-1)}(x) = a\ \tan(x) - b$

and

$X = a\ \tan[\delta t + \arctan(\frac{x+b}{a})] - b = f^{(-1)}(\delta t + f(x)) = F^{((\delta t))}(1, x).$

We have

$\tan(\alpha+\beta) = \frac{\tan(\alpha)+\tan(\beta)}{1-\tan(\alpha)\tan(\beta)}$

and

$\arctan(u) + \arctan(v) = \arctan[\frac{u+v}{1-uv}],$

so

$$X = a\ \frac{\tan(\delta t) + (\frac{x+b}{a})}{1 - \tan(\delta t)(\frac{x+b}{a})} - b$$

$$= \frac{(a^2 \tan(\delta t) + a(x+b) - b(a-(b+x)\tan(\delta t))}{a-(b+x)\tan(\delta t)}$$

$$= \frac{(a^2+b^2)\tan(\delta t) + (a + b\tan(\delta t))x}{a - \tan(\delta t)b - \tan(\delta t)x},$$

the expression in Granau.

Note that if you have the formal group law for an inverse pair of functions $h(x)$ and $h^{(-1)}(x)$ with $h(0) = 0$

$$FGL_h(u,v) = h^{(-1)}[h(u) + h(v)] = F(h(u), v),$$

then you can relate it to a shifted FGL not necessarily vanishing at the origin

$$X = F^{((\delta t))}(x) = F(\delta t, x) = f^{(-1)}(\delta t + f(x)) = FGL_f(f^{(-1)}(\delta t), x)$$

to quickly express X in terms of δt and x. For $h(x) = \arctan(x)$,

$$FGL_{atan}(u,v) = \tan[\arctan(u) + \arctan(v)] = \frac{u+v}{1-uv}.$$

(For a short discussion of FGLs, see the webpage "Formal Groups and Where to Find Them" by Walker.)

With

$f(x) = \arctan(\frac{x+b}{a})$ and $f^{(-1)}(x) = a\ \tan(x) - b$,

$$X = F^{((\delta t))}(x) = F(\delta t, x) = f^{(-1)}(\delta t + f(x))$$

$$= a\ \tan[\ \delta t + \arctan(\frac{x+b}{a})\ ] - b$$

$$= FGL_f(f^{(-1)}(\delta t), x)$$

$$= a\ \tan[\ \arctan(\tan(\delta t)) + \arctan(\frac{x+b}{a})\ ] - b$$

$$= a\ FGL_{atan}[\ \tan(\delta t),\ \frac{x+b}{a}\ ] - b$$

$$= a\ \frac{u+v}{1-uv} - b$$

with $u = \tan(\delta t)$ and $v = \frac{x+b}{a}$.

The solution for the result of the compositional iteration is written by Frege as the linear fractional, of Moebius, transformation

$$X = \frac{A+Bx}{C+Dx}.$$

Compare this with eqn, (1.21) on pg. 16 of "Part I. The simple Galton-Watson process: Classical approach" by Gerold Alsmeyer and with eqn. (7.4) on pg. 9 of "The Theory of Branching Processes" by Harris.

Granau-Frege's equations for a, b, and $\tan(n) = \tan(\delta t)$ in terms of $A, B, C,$ and D can be re-expressed in terms of the zeros of

$$y(x) = \alpha x^2 + \beta x + \gamma = Dx^2 - i(C-B)x + A$$

with $D = \alpha$, $\gamma = A$, and $\beta = i(B-C)$. Then

$$a = \frac{\sqrt{-(C-B)^2 - 4AD}}{2D} = \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$$

$$b = \frac{C-B}{2D} = \frac{i\beta}{2\alpha},$$

$ib \pm a = \frac{-\beta \pm \sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha}$,

$$\tan(n) = \tan(\delta t) = \frac{2D}{B+C} \frac{\sqrt{-(C-B)^2 - 4AD}}{2D}$$

$$= \frac{2\alpha}{B+C} \frac{\sqrt{\beta^2 - 4\alpha\gamma}}{2\alpha},$$

$C - B = 2D\ b,$

$C + B = \frac{\sqrt{-(C-B)^2 - 4AD}}{2\tan(\delta t)}.$

Note that $\ln(X) = \ln[\frac{A+Bx}{C+Dx}]$ can be related to some formulas in A008292 involving the hyperbolic FGL, a generalized tangent-arctangent FGL.

Let's use Frege's trivial first example as a second check on the formalism. In my notation,

with $f(x) = \ln(x)/\ln(b)$ and $f^{(-1)}(x) = b^x = e^{x\ \ln(b)}$,

$F(1,x) = f^{(-1)}(1 + f(x)) = \exp[\ \ln(b) + \ln(x)\ ] = b\ x$.

Then $X = f^{(-1)}[n + f(x)] = b^{n+\ln(x)/\ln(b)} = F^{((n))}(1,x) = b^n\ x$.

With $h(x) = \ln(1+x)$ and $h^{(-1)}(x) = e^x - 1$,

$FGL_{\ln}(u,v) = h^{(-1)}[h(u) + h(v)] = \exp[\ \ln(1+u) + \ln(1+v)\ ] - 1 = u + v + uv$.

Then $h(x) = \ln(b)\ f(1+x)$ and $h^{(-1)}(x) = f^{(-1)}(x/\ln(b)) - 1$.

Conversely, $f(x) = h(x-1)/\ln(b)$ and $f^{(-1)}(x) = h^{(-1)}(x\ln(b)) + 1$, so

$X = f^{(-1)}[\ \delta t + f(x)\ ] = h^{(-1)}[(h(h^{(-1)}(\delta t\ln(b))) + h(x-1)\ ] + 1$

$= [FGL_{ln}(u,v) + 1]\ |_{u=h^{(-1)}(\delta t\ln(b)),\ v=x-1}$

$$= [FGL_{ln}(u,v) + 1] \,|_{u=b^{\delta t}-1,\; v=x-1}$$

$$= (b^{\delta t} - 1) + (x-1) + (b^{\delta t} - 1)(x-1) + 1 = b^{\delta t}\, x,$$

In agreement with the briefer straightforward computation.

Letting

$$G(X,x) = e^{\delta t(X,x))} = e^{f(X)-f(x)} = \exp[\int_x^X \frac{du}{g(u)}],$$

then

$$G(X,x) = G(X,y) \cdot G(y,x),$$

and we have the makings of a Green's function and path integration over an action, which we could morph with the imaginary number $(\delta t \to i\ \delta t)$ into phase factors, as shown in earlier sections.

Related stuff: (with some redundancy)

On the physicists' approach to the RG in quantum and statistical mechanics:

"Phase transitions and renormalization group: from theory to numbers" by Zinn-Justin (paper)

"Phase Transitions and Renormalization Group" by Zinn-Justin (book)

"Scaling, Self-similarity, and Intermediate Asymptotics" by Rosenblatt

"Lectures on Phase Transitions and the Renormalization Group" by Goldenfeld

"Renormalization Group theory and Variational calculations for propagating fronts" by Chen and Goldenfeld

“Numerical renormalization group calculations for similarity solutions and traveling waves” by Chen and Goldenfeld

“The renormalization group and critical phenomena” by WIlson (Nobel lecture)

“Quantum Field Theory Vols, I, II, and III” by Zeidler (classical and Hopf algebra approach to the renormalization}

“Why the renormalization group is a good thing” by Weinberg

“Mathematical developments in the rise of Yang-Mills gauge theories” by Koberinsk. See pg. 9 on Gell-Mann and Low, Pg..18 Sec.3.2 Development of the renormalization group, and pg. 27 4.2 The success of formal analogies with statistical mechanics.

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“The renormalization group and the ϵ-expansion” by Wilson and Kogut (p. 179, differential equation)

“The Critical Point” by Domb, in particular, section “1.2 Renormalization Group: Respectability{ (p. 29) and chapter “7 Renormalization Group” (p. 261)

“Theories of Matter: Infinities and Renormalization” by Kadanof, in particular, “Section 6.6 The Wilson revolution}” (p.41).

“Wilson’s renormalization group: a paradigmatic shift” by Brézin

“Quantum Electrodynamics at Small Distances” by Gell-Mann and Low

“A Critical History of Renormalization” by Kerson Huang

“Renormalization and Effective Field Theory” by Costello, pg. 124 introduces the vector field associated with a RG flow

On complex dynamics and functional iteration / composition:

“Iteration Theory and its Functional Equations” edited by Liedl, Reich, and Targonski

“Convolution Polynomials” by Knuth

“A History of Complex Dynamics: From Schröder to Fatou to Julia” by Alexander

“Farey Curves” by Buff, Henriksen, and Hubbard

“Newton’s method and fractals” by Burton

“An introduction to Julia and Fatou sets” by Sutherland

“Newton’s Method as a Dynamical System” by Ruckert

“Topology for the basins of attraction of Newton’s method in two complex variables” by Roeder

“Mandelbrot set” Wikipedia

“Families of Rational Maps and Iterative Root-Finding Algorithms” by McMullen

Book review by Perez of “Early days in complex dynamics: a history of complex dynamics in one variable during 1906–1942”, by Alexander, Iavernaro, and Rosa

“Translation equation and Sincov’s equation – A historical remark” by Granau

“A remark on Sincov’s functional equation” by Gronau

On the RG and CD:

“Evolution profiles and functional equations” by Curtright and Zachos

“Chaotic Maps, Hamiltonian Flows, and Holographic Methods” by Curtright and Zachos

“Logistic Map Potentials” by Curtright and Veitia

“Renormalization Group Functional Equations” by Curtwright and Zachos

“Scaling, Self-similarity, and Intermediate Asymptotics” by Barenblatt

“Lectures on Phase Transitions and the Renormalization Group “ by Goldenfeld, section 9.5 RG in Differential Form on pg. 256

On variational principles and classical mechanics:

“The principle of least action” by Feynman

“Conceptual Approaches to the Principles of Least Action” by Danielsson

“Hamilton’s principle: why is the integrated difference of kinetic and potential energy Minimized?” by Rojo

“The Principle of Least Action in Dynamics” and Manton

“Fermat’s Principle and the Geometric Mechanics of Ray Optics” by Holm

“Principle of Least Action” by Manoj K. Harbola (slides)

Web pages on Fermat’s principle, on Abbreviated Action, on Maupertuis’ Action, on Hamilton’s Principle, on the Hamilton-Jacobi Equation by Fowler

Hamilton-Jocobi Equation (Wikipedia)

“The Variational Principles of Mechanics” by Lanczos

“Variational Principles in Classical Mechanics” by Cline

“Mechanics and the Foundations of Modern Physics” by Helliwell and Sahakian

“Mathematical Methods of Classical Mechanics” by Arnold

“Geometric Numerical Integration” by Hairer, Lubich, and Wanner

On the Hopf algebraic approach to the RG:

“Hopf algebras in renormalization theory: Locality and Dyson-Schwinger equations from Hochschild cohomology” by Bergbauer and Kreimer

“Renormalization in quantum field theory and the Riemann-Hilbert problem II: the β-function, diffeomorphisms and the renormalization group” by Connes and Kreimer

“Hopf Algebras, Renormalization and Noncommutative Geometry” by Connes and Kreimer

“Renormalization & Renormalization Group” by Kreimer

“Renormalization Group as a Probe for Dynamical Systems” by Sarkar and Bhattacharjee

“QFT and its discontents: a blog” by Predrag Cvitanovic

"On algebraic structures of numerical integration on vector spaces and manifolds" by Lundervold and Munthe-Kaas

More on the underlying differential equations, algebra, and combinatorics:

"Compositional Inverse Pairs, the Inviscid Burgers-Hopf Equation, and the Stasheff Associahedra" by Copeland

The OEIS entries above repeat some of the refs above and supply more as do the MathOverflow questions referenced in the entries. My blog "Shadows of Simplicity" also contains many relevant posts, some directly related to the above, others, tangentially.

Miscellaneous:

"Applications of Lie Groups to Differential Equations" by Olver

For an extended discussion of coupling constants in physics, see "Theories of Matter: Infinities and Renormalization" by Kadanoff

"Lectures on the Theory of Group Properties of Differential Equations" by Ovsyannikov

"Analytical Form of Differential Equations" by Brjuno

'From small divisors to Brjuno functions" by Marmi (see pg. 24)

"Branching processes 1. Galton-Watson processes" lecture notes by Steve Lalley, Univ. of Chicago, on the Galton-Watson process (eqn. (5) on pg. 3 and eq.(22) on pg. 12)

"Universal Scaling Behaviour for Iterated Maps in the Complex Plane" by N. S. Manton and M. Nauenberg

"Lecture Notes on Dynamical Systems, Chaos, and Fractal Geometry" by Goodson

"Renormalization group as as a probe of dynamical systems" by Sarkar and Bhattacharjee

The RG flow tangency condition appears also in equations 2.4-2.6 and 2.21 of "Renormalization-group method for reduction of evolution equations: invariant manifolds and envelopes" by Ei, Fujii, and Kunihiro.

"Introduction to Partial Differential Equations" by Olver

“The Calculus of Variations” by Olver

“Manifolds, Tensor Analysis, and Applications” by Marsden, Ratiu, and Abraham (3’rd edition), in particular Chapter 4 Vector Fields and Dynamical Systems

“Introduction to Mechanics and Symmetry” by Marsden and Ratiu (2’nd edition)

“On Symplectic Reduction in Classical Mechanics” by Butterfield

“Renormalization: From Lorentz to Landau (and Beyond)” by Brown (editor)

“The conceptual foundations and the philosophical aspects of renormalization theory” by Cao and Schweber

“The Triumph and Limitations of Quantum Field Theory” by Gross

“Burgers’s-Korteweg-De Vries Equation for Viscoelastic Rods and Plates” by Nariboli and Sedov

“On tree hook length formulae, Feynman rules and B-series” by Bradley Robert Jones

“Introduction to Smooth Manifolds” by John Lee

“Modern Geometry--Methods and Applications: Part I, The Geometry of Surfaces, Transformations, and Fields” by Dubrovin, Fomenko, and Novikov (2’nd edition) : Maupertuis’ Principle p.345. Geodesic metric proportional to E -U on p.346. Fermat’s principle 347. Lagrange surfaces, HJE p. 367.

For some early presentations of formal group laws, see https://mathoverflow.net/questions/352842/nascent-formal-group-law

See pgs. 8 and 9 of “From Abel’s heritage: Transcendental objects in algebraic geometry and their algebrazation” by Catanese for an example of an Abel equation and reduction of integrals

“The Biggest Ideas in the Universe: 11. Renormalization ” video by Sean Carroll, very good (see some other vids in YouTube library)

“6 Lectures on QFT, RG, and SUSY” by Hollywood

“A hint of renormalization” very good, and more adv “An intro to nonperturbative renormalization group” by Delamotte

“Regularization, renormalization, and dimensional analysis: Dimensional regularization meets freshman E&M” and another “Regularization, renormalization, and dimensional analysis” by Olness and Scalise

Neumaier, “Renormalization without infinities”

Li “Intro to Renormalization in Field Theory” power counting

Butterfield and Bonatta, “Renormalization for Philosophers” good

“Analysis of a renormalization group method and normal form theory for perturbed ordinary differential equations” by DeVille, Harkin, Holzec Josic, Kaper

“Bogoliubov Renormalization Group and Symmetry of Solution in Mathematical Physics” by Vladimir F. Kovalev and Dmitrij V. Shirkov has on pg. 21 Burgers’ equation soln for initial boundary value f(0,x). This is derived from the heat equation. Look also at other paper ? which gives an example of CLT from iterative procedure and compare/merge perspectives. https://arxiv.org/pdf/hep-th/0001210.pdf

“Functional self-similarity and renormalization group symmetry in mathematical physics” by Vladimir F. Kovalev and Dmitrij V. Shirkov

https://www.physicsoverflow.org/37454/pedagogical-introduction-st%C3%BCckelberg-renormalization-group?show=37454#q37454

The Bogoliubov Renormalization Group (Second English printing) by D.V. Shirkov, pg. 8 contains the Gell-Mann Low Lee solution and good history

Conceptual Foundations of Quantum Field Theory by Cao

“Symmetries of Integro-Differential Equations” by Y.N. Grigoriev N.H. Ibragimov V.F. Kovalev S.V. Meleshko, saved under Kovalev Burgers equations

Quantum Field Theory II: Quantum Electrodynamics by Zeidler, on renormalization group

“What is the event in history where iterated functions became appropriate for modeling physics?” Physics StackExchange question posted by Geisler

“Frege’s Habilitationsschrift: Magnitude, Number and the Problems of Computability” by Gastaldi

“Structural features in Ernst Schröder's’s work. Part I” by Bondoni

“Structural features in Ernst Schröder's’s work. Part II” by Bondoni, in which the footnote on p. 282 begins:

Yet in 1884 Koenigs wrote in his investigations, *Mr. Schröder met with a functional equation, from which one may deduce Abel's by taking the logarithm of the two members*. To solve the Abel Equation or Schröder's is then the same problem [Koe84, p. 4]. The relation between the Schröder Equation and Abel's was also stressed by Pincherle: . . . .

"Fractional Iteration and Functional Equations for Functions Analytic in the Unit Disk" by Elin, Goryainov, Reich, and Shoikhet

"Renormalisation Group, Function Iterations, and Computer Algebra" by Caprasse

"Newton's Method and Complex Dynamical Systems" by Haeseler and Peitgen