The Creation Op $\mathfrak{D} = q(z) + g(z)\partial_z$*:*

Scaled Flow and Operator Identities

by Tom Copeland, Los Angeles, Ca., 2/2/2022

In the West, the dragon symbolizes greed--a wicked creature wreaking havoc and hoarding gold and maidens it can never use. In the East, the dragon symbolizes freedom--an imposing creature who can swim, run, and fly, navigating the borders between order, chaos, and growth.

These notes are but one legacy of admirable dragons ... Scherk, Graves, Cayley, Hermite, Appell, Sheffer, Pincherle, Born, Heisenberg, Weyl, Witt, Virasoro, Kac, Schwarz …

In this set of notes, I'll characterize the exponentiated action on analytic functions of the first-order differential op

$$\mathfrak{D} = q(z) + g(z)\partial_z$$

where, in general, $q(z)$ and $g(z)$ are arbitrary complex functions analytic at the point $z_p = z$ in the complex plane, and ∂_z is the partial derivative w.r.t. z. (Mnemonics underlie the notation; $g(z)$ defines a Lie infinitesimal **g**enerator and $q(z)$ leads to a **q**uotient factor in the primary operational identity.) I'll then relate this characterization to operational identities found in "Enumerative geometry, tau-functions and Heisenberg–Virasoro algebra" by Alexandrov related to algebras associated with Weyl, Heisenberg, Witt, Virasoro, Kac, and Schwarz.

The primary operational formula is the exponential action on an arbitrary analytic function $H(z)$

$$e^{t\mathfrak{D}}\ H(z) = e^{t(g(z)\partial_z + q(z))}H(z) = \frac{V(f(z)+t)}{V(f(z))}\ H(f^{(-1)}(f(z)+t))$$

$$= \frac{V(f(z)+t)}{V(f(z))}\ H(FL(z,t))$$

with $\omega = f(z)$ an analytic curve with local compositional inverse $z = f^{(-1)}(\omega)$,

$$g(z) = \frac{1}{\partial_z f(z)},$$

$e^{t(g(z)\partial_z + q(z))}1 = \frac{V(f(z)+t)}{V(f(z))}$,

and

$$e^{t\ g(z)\partial_z}\ z = f^{(-1)}(f(z)+t) = FL(z,t).$$

$V(z)$ is further characterized by

$v(z) = q(f^{(-1)}(z))$,

$v(z) = \partial_z \ln[V(z)]$,

and

$$V(z) = e^{\int^z\ v(\omega)\ d\omega},$$

where the indefinite integral represents a primitive of $v(z)$, where by primitive I mean a function of z devoid of any summand independent of z, whose derivative gives $v(z)$. The log derivative will filter out any legit lower limit of a definite integral of $v(z)$ as well as any overall scalar factor C independent of z;

with $\int_{lb}^{z}\ v(\omega)\ d\omega = Pv(\omega)\ |_{\omega=lb}^{\omega=z} = Pv(z) - Pv(lb)$, then

$V_{lb}(z) = e^{Pv(z)-Pv(lb)} = e^{-Pv(lb)}\ e^{Pv(z)}$ and

$v(z) = \partial_z\ \ln(C\ V_{lb}(z)) = \partial_z\ \ln(\bar{C}\ e^{Pv(z)}) = \partial_z\ Pv(z)$,

where $\bar{C} = C\ e^{-Pv(lb)}$ is independent of z.

This invariance w.r.t. to the factor C and a lower limit of integration is reflected in the ratio

$\frac{V(f(z)+t)}{V(f(z))} = \frac{e^{Pv(f(z)+t)}}{e^{Pv(f(z))}}$ in the primary formula. (This invariance has the scent of a Schwarzian derivative about it with connections to the KdV, Virasoro group, and the heat/diffusion/Airy equation.)

The function

$$\widetilde{H}(x,t) = \frac{V(f(z)+t)}{V(f(z))}\ H(FL(z,t))$$

is the formal solution of the p.d.e.

$$\partial_t \, \widetilde{H}(x,t) = \mathfrak{D} \, \widetilde{H}(x,t)$$

with initial condition $\widetilde{H}(x,0) = H(x)$.

An instance of the creation / raising op eqn

$$e^{t(g(z)\partial_z + q(z))} 1 = \frac{V(f(z)+t)}{V(f(z))}$$

is eqn. 269 on p. 75 of "On Topological 1D Gravity. I by Jian Zhou, related to the Hermite polynomials of OEIS A099174, which in turn are related to A344678. Both are related to the Heisenbeg-Weyl algebra, in which the commutator $[L,R]] = LR - RL = 1$ and its generalization $[f(L),R]] = \frac{df(L)}{dL} = f'(L)$ play the definitive roles. (The roles of R and L can be switched. Charles Graves and Pincherle made much use of this commutator as well as Lie, Born, Heisenberg, Jordan, Wey, and others.) One realization of the pair of ladder ops--the raising / creation op R and the lowering / destruction / annihilation op L--is the pair $L = \partial_z$ and $R = z$; another, $L = \partial_z$ and $R = z + \partial_z$, for the family of Hermite polynomials above. $g(z)\partial_z$ and and $f(z)$ satisfy the basic commutator relation as well.

Notation: where there might be some confusion about whether an equation involves equating operators or equating the action/results of operations on a function, I'll use $\odot$ to indicate the expression to its left is an op, not a function, so the equation in that case is relating ops equivalent in action on a given class of functions. Of course, always keep in mind that this equivalence can easily vanish with a change in the class of functions acted upon.

Review of flow functions and iterated Lie derivatives/vector fields:

Given the inverse pair

$\omega = f(z)$ and $z = f^{(-1)}(\omega)$

analytic about the origin (z,ω)=$(0,0)$ with $f(0) = 0$, let

$$g(z)\partial_z = \frac{1}{f'(z)}\frac{\partial}{\partial z} = \frac{\partial}{\partial_z f(z)} = \frac{\partial}{\partial \omega}.$$

I'll refer to $g(z)\partial_z$ as an infinitesimal generator (IG), but it is also referred to as a Lie infinitesimal generator, Lie vector derivative, or Lie vector (slighting earlier work by Charles Graves), or simply vector.

Then for small enough $|z|$ and $|t|$, we can generate a flow function $FL(,t)$ via iterating the IG as

$$e^{tg(z)\partial_z}\ z = e^{t\frac{\partial}{\partial\omega}}\ f^{(-1)}(\omega) = f^{(-1)}(\omega+t) = f^{(-1)}(f(z)+t) = FL(z,t),$$

implying several identities or properties:

$$\partial_t e^{tg(z)\partial_z}\ z = g(z)\partial_z\ e^{tg(z)\partial_z}\ z,$$

or

$\partial_t FL(z,t) = g(z)\partial_z FL(z,t)$, an ***Evolution p.d.e.*,**

$FL(z,t) = f^{(-1)}(f(z)+t))$, a ***Flow function***,

$FL(z,0) = f^{(-1)}(f(z)) = z$, ***Identity property***,

$FL(0,t) = f^{(-1)}(t)$, ***Inversion identity / orbit of the flow***,

$\partial_t FL(z,t) - g(z)\partial_z FL(z,t) = 0$, ***Tangent identity***,

For sufficiently small $|z|$, $|s|$, and $|t|$, or under analytic continuation, we have the

Group property for the flow function

$$FL(FL(z,s),t) = f^{(-1)}[f(f^{(-1)}(f(z)+s))+t] = f^{(-1)}(f(z)+s+t) = FL(z,s+t)$$

so

$FL^{(-1)}(z,t) = FL(z,-t) = e^{-tg(z)\partial_z}\ z$, ***Flow function inversion identity***,

An associated o.d.e. and the inverse function derivative identity are easily developed:

$$\frac{\partial}{\partial t}\ FL(z,t)\ |_{t=0} = \frac{\partial}{\partial t}\ f^{(-1)}(f(z)+t)\ |_{t=0} = (f^{(-1)})'(f(z))$$

but also

$$\frac{\partial}{\partial t}\ FL(z,t)\ |_{t=0} = \frac{\partial}{\partial t}\ e^{tg(z)\partial_z}\ z\ |_{t=0} = g(z) = \frac{1}{f'(z)},$$

so

$(f^{(-1)})'(f(z)) = g(z) = \frac{1}{f'(z)}$, ***Inverse function theorem***,

and upon substitution

$(f^{(-1)})'(\omega)) = \frac{d}{d\omega}\ f^{(-1)}(\omega) = g(f^{(-1)}(\omega))$, ***Autonomous o.d.e.***

The inverse function formula follows easily also from the chain rule via

$\partial_z z = \partial_z f^{(-1)}(f(z))$, giving $1 = (f^{(-1)})'(f(z)) \cdot f'(z)$.
In addition,

$$g(FL(z,t)) = (f^{(-1)})'(f(FL(z,t))) = (f^{(-1)})'(f(z)+t) = \frac{\partial}{\partial t}FL(z,t)$$

$$= g(z)\frac{\partial Fl(z,t)}{\partial z}.$$

Cutting out the intermediate steps, we obtain the

Flow function diff eq / tangency property

$\frac{\partial}{\partial t}FL(z,t) = g(z)\frac{\partial Fl(z,t)}{\partial z} = g(FL(z,t))$,

encompassing a tangency condition with respect to the vector $\partial_t - g(z)\partial_z$.

Note that several relations hold even for $f(0) \neq 0$. See OEIS A145271 for more info.

Action of the exponential map of the IG,
i.e., group action of the IG, on an analytic function $H(z)$:

$$e^{tg(z)\partial_z}\ H(z) = e^{t\frac{\partial}{\partial\omega}}\ H(f^{(-1)}(\omega)) = H(f^{(-1)}(\omega+t))$$

$$= H(f^{(-1)}(f(z)+t))) = H(FL(z,t)),$$

and we see that this is the generalized shift operator introduced by Charles Graves in the 1850s, but implicit in the Taylor series via a change of variables..

Then

$$e^{tg(z)\partial_z}\ z\ e^{-tg(z)\partial_z}\ H(z) = e^{tg(z)\partial_z}\ z\ H(FL(z,-t))$$

$$= FL(z,t)\ H(FL(FL(z,t),-t)) = FL(z,t)H(z),$$

so for action on analytic functions, we can identify

$$e^{tg(z)\partial_z}\ z\ e^{-tg(z)_z}\ \odot = f^{(-1)}(f(z)+t)\ \odot = FL(z,t)\ \odot$$

and, conversely,

$$e^{-tg(z)\partial_z}\ z\ e^{tg(z)\partial_z}\ \odot = f^{(-1)}(f(z)-t)\ \odot = FL(z,-t) = FL^{(-1)}(z,t)\ \ \odot\ .$$

Consistently,

$$e^{tg(z)\partial_z}\ z\ e^{-tg(z)\partial_z} z = e^{tg(z)\partial_z}\ z\ FL(z,-t)$$

$$= FL(z,t)FL(FL(z,t),-t) = FL(z,t)FL(FL^{(-1)}(z,t),t) = FL(z,t)z,$$

and

$$e^{tg(z)\partial_z}\ z^n = (e^{tg(z)\partial_z}\ z\ e^{-tg(z)\partial_z})^{n-1} e^{tg(z)\partial_z}\ z$$

$$= (FL(z,t))^{n-1} FL(z,t) = (FL(z,t))^n.$$

The exponentiated creation op $\mathfrak{D} = q(z) + g(z)\partial_z$ ***:***

Denote the result of exponentiated action on an analytic function $H(z)$ as

$$e^{t(g(z)\partial_z + q(z))} H(z) = \tilde{H}(z,t),$$

with the associated evolution equation

$$\frac{\partial}{\partial t}\ \tilde{H}(z,t) = (g(x)\frac{\partial}{\partial z} + q(z))\ \tilde{H}(z,t).$$

Transforming using the local inverse of $f(z)$ gives

$$e^{t(\partial_\omega + q(f^{(-1)}(\omega)))}\ \hat{H}(\omega)\ = e^{t(\partial_\omega + v(\omega))}\ \hat{H}(\omega) = K(\omega,t)$$

with

$$g(z)\partial_z = \frac{1}{f'(z)}\partial_z = \frac{\partial}{\partial f(z)} = \frac{\partial}{\partial \omega},$$

$$v(\omega) = q(f^{(-1)}(\omega)),$$

$$\hat{H}(\omega) = H(f^{(-1)}(\omega)) = H(z),$$

and

$$K(\omega,t) = \tilde{H}(f^{(-1)}(\omega),t) = \tilde{H}(z,t)$$

when $(z,\omega) = (f^{(-1)}(\omega), f(z))$ for the local analytic inverse pair of functions.

The associated evolution p.d.e. is

$$\frac{\partial}{\partial t}\ K(\omega,t) = (\frac{\partial}{\partial \omega} + v(\omega))\ K(\omega,t) = V^{-1}(\omega)\ \frac{\partial}{\partial \omega}\ V(\omega)\ K(\omega,t)$$

with

$$v(\omega) = \frac{d}{d\omega}\ln[V(\omega)],$$

so

$$\frac{\partial}{\partial t}\ V(\omega)K(\omega,t) = \frac{\partial}{\partial \omega}V(\omega)\ K(\omega,t),$$

or

$$\frac{\partial}{\partial t}\ L(\omega,t) = \frac{\partial}{\partial \omega}\ L(\omega,t)$$

with the general solution, with the initial condition

$L(\omega,0)=\tilde{L}(\omega)=V(\omega)K(\omega,0)$, given by

$$L(\omega,t)=e^{t\partial_w}\tilde{L}(\omega)=\tilde{L}(\omega+t)=V(\omega+t)K(\omega+t,0).$$

Then, with the the integrating factor

$$V(\omega)=e^{\int\ v(\omega)\ d\omega},$$

the solution of

$$\frac{\partial}{\partial t}\ K(\omega,t)=(\frac{\partial}{\partial\omega}+v(\omega))\ K(\omega,t)=V^{-1}(\omega)\ \frac{\partial}{\partial\omega}\ V(\omega)\ K(\omega,t)$$

with the initial condition

$$K(\omega,0)=\tilde{L}(\omega)$$

is

$$K(\omega,t)=\frac{V(\omega+t)}{V(\omega)}\ \tilde{L}(\omega+t),$$

which satisfies

$$e^{t(\partial_\omega+q(f^{(-1)}(\omega)))}\ \tilde{L}(\omega)\ =e^{t(\partial_\omega+v(\omega))}\ \tilde{L}(\omega)=K(\omega,t).$$

Note that only the primitive of $v(\omega)$ plays a role in the solution, as discussed in more detail in the intro section, since any constant of integration factors out, in the determination of $V(\omega)$, as an exponential of the constant which would ultimately be removed by the ratio $\frac{V(\omega+t)}{V(\omega)}$ in the solution.

With $\tilde{L}(\omega)=1$, this gives

$$e^{t(\partial_\omega+q(f^{(-1)}(\omega)))}\ 1=e^{t(\partial_\omega+v(\omega))}\ 1$$

$$=e^{tV^{-1}(\omega)\partial_\omega V(\omega)}1=\frac{V(\omega+t)}{V(\omega)},$$

which transforms back to

$e^{t(g(z)\partial_z+q(z))}1 = \frac{V(f(z)+t)}{V(f(z))}$

and, with arbitrary analytic $\tilde{L}(\omega)$ but $v(\omega) = 0$,

$e^{t\partial_\omega}\ \tilde{L}(\omega) = \tilde{L}(\omega+t),$

which, with $\tilde{L}(\omega) = H(f^{(-1)}(\omega))$, transforms back to

$e^{tg(z)\partial_z(\omega)}\ H(z) = H(f^{(-1)}(f(z)+t)).$

Putting it all together, the full back-transformation is

$e^{t(g(z)\partial_z+q(z))}H(z) = \frac{V(f(z)+t)}{V(f(z))}\ H(f^{(-1)}(f(z)+t)) = \frac{V(f(z)+t)}{V(f(z))}\ H(FL(z,t)).$

Spot check:

With $v(\omega) = \omega$, then $V(\omega) = \ e^{\frac{\omega^2}{2}}$, and with the initial condition $\tilde{L}(\omega) = 1$,

$K(\omega,t) = \frac{e^{\frac{(\omega+t)^2}{2}}}{e^{\frac{\omega^2}{2}}} = e^{\frac{t^2}{2}}e^{\omega t}.$

allegedly satisfies

$e^{t(\omega+\partial_\omega)}\ K(\omega,0) = K(\omega,t) = e^{t(\omega+\partial_\omega)}\ 1 = e^{\frac{t^2}{2}}e^{\omega t}$

with the associated evolution p.d.e.

$\frac{\partial}{\partial t}\ K(\omega,t) = (\omega+\partial_\omega)\ K(\omega,t).$

The raising op for the family of Hermite polynomials with the e.g.f. $e^{tHer.(\omega)} = e^{\frac{t^2}{2}}\ e^{\omega t}$ is $R = \omega + \partial_\omega,$ so

$$e^{tR}\ 1 = e^{t(\omega+\partial_\omega)}\ 1 = e^{tHer.(\omega)} = e^{\frac{t^2}{2}}\ e^{\omega t};$$

consequently, the solution of the evolution equation with initial condition $K(\omega,0)=1$

should be this e.g.f., i.e., $K(\omega,t) = e^{\frac{t^2}{2}}\ e^{\omega t}$.

Check:

$$\partial_t e^{\frac{t^2}{2}}\ e^{\omega t} = (t+\omega)e^{\frac{t^2}{2}}\ e^{\omega t}$$

and, consistently,

$$(\omega+\partial_\omega)e^{\frac{t^2}{2}}\ e^{\omega t} = (\omega+t)e^{\frac{t^2}{2}}\ e^{\omega t}.$$

Characterization as a creation / raising op

$$e^{t(\partial_\omega + q(f^{(-1)}(\omega)))}\ 1 = e^{t(\partial_\omega+v(\omega))}\ 1 = e^{tV^{-1}(\omega)\partial_\omega V(\omega)} = \frac{V(\omega+t)}{V(\omega)}$$

implies

$$\partial^n_{t=0}\ \frac{V(\omega+t)}{V(\omega)} = (\partial_\omega + q(f^{(-1)}(\omega)))^n\ 1$$

$$= (\partial_\omega + v(\omega))^n\ 1 = (V^{-1}(\omega)\partial_\omega V(\omega))^n\ 1,$$

that is, the op

$$R = \partial_\omega + q(f^{(-1)}(\omega)) = \partial_\omega + v(\omega) = V^{-1}(\omega)\partial_\omega V(\omega)$$

is a raising op for the Taylor series coefficients $T_n(\omega)$ of the series expansion in t of $\frac{V(\omega+t)}{V(\omega)}$,i.e.,

$$\partial^n_{t=0}\ \frac{V(\omega+t)}{V(\omega)} = T_n(\omega) = R\ T_{n-1}(\omega) = R^n\ 1$$

with $T_0(\omega)=1$.

This characterization as a creation op is preserved in transforming back to our initial coordinates

$$e^{t(g(z)\partial_z + q(z))} 1 = \frac{V(f(z)+t)}{V(f(z))}.$$

One of the simplest examples is the raising op $R = z + \partial_z$ for the probabilist's Hermite (or Chebyshev) Appell Sheffer polynomials, as mentioned above (cf. A344678).

A group property of the V quotient

For another method of derivation and some history on these types of ops, see "Boson normal ordering via substitutions and Sheffer-type polynomials" by Blasiak, Horzela, Penson, Duchamp, and Solomon. They present an easily derived group property of the V quotient. (Other refs with different characterizations and associated combinatorics are given in the Related Stuff section below.)

Equations 4 on p. 2 of Blasiak et al. are consistent with the formulas here. Their first group property has already been presented above, and the second, in their notation,

$$g(\lambda + \theta, x) = g(\lambda, x) g(\theta, T(\lambda, x))$$

corresponds to

$$\frac{V(f(z)+t+s)}{V(f(z))} = \frac{V(f(z)+t)}{V(f(z))} \frac{V(f(FL(z,t))+s)}{V(f(FL(z,t)))} = \frac{V(f(z)+t)}{V(f(z))} \frac{V(f(z)+t+s)}{V(f(z)+t)},$$

or, with $RV(z,t) = \frac{V(f(z)+t)}{V(f(z))}$,

$$RV(z, t+s) = RV(z,t) RV(FL(z,s), t).$$

Equation 3 on p. 2 of Blasiak et al.

$$\partial_\lambda \, g(\lambda, x) = v(T(\lambda, x), x) g(\lambda, x)$$

corresponds, with the obvious confusion of notation, to

$$\partial_t \ln[\frac{V(f(z)+t)}{V(f(z))}] = \frac{V'(f(z)+t)}{V(f(z)+t)} = v[f(z)+t] = q[FL(z,t)].$$

Conjugating ops

Reprising,

$$e^{tg(z)\partial_z}\ H(z) = H(FL(z,t)),$$

so

$$e^{tg(z)\partial_z}\ H(z)\ e^{-tg(z)\partial_z} M(x) = e^{tg(z)\partial_z}\ H(z)\ M(FL(z,-t))$$

$$= H(FL(z,t))\ M(FL(FL(z,t),-t)) = H(FL(z,t))\ M(z),$$

and abstracting, we have, for the conjugation, the operator equivalence

$$e^{tg(z)\partial_z}\ H(z)\ e^{-tg(z)\partial_z} \odot = H(FL(z,t))\ \odot .$$

Similarly,

$$e^{t(\partial_\omega + v(\omega))}\ H(\omega)\ e^{-t(\partial_\omega + v(\omega))}\ M(\omega)$$

$$= e^{t(\partial_\omega + v(\omega))}\ H(\omega)\ M(\omega - t)\frac{V(\omega-t)}{V(\omega)}$$

$$= H(\omega+t)\ M(\omega+t-t)\frac{V(\omega+t-t)}{V(\omega+t)}\frac{V(\omega+t)}{V(\omega)}$$

$$= H(\omega+t)\ M(\omega),$$

so abstracting, the conjugation is equivalent to

$$e^{t(\partial_\omega + v(\omega))}\ H(\omega)\ e^{-t(\partial_\omega + v(\omega))} \odot = H(\omega+t)\ \odot .$$

Returning to our original local variable/coordinate patch $(z,\omega) = (f^{-1}(\omega), f(z))$,

$$e^{t(g(z)\partial_z + q(z))}\ H(f(z))\ e^{-t(g(z)\partial_z + q(z))}\ M(f(z)) = H(f(z)+t)\ M(f(z))$$

transforms,

with $H(\omega)=\widetilde{H}(z)=\widetilde{H}(f^{(-1)}(\omega))$ and $M(\omega)=\widetilde{M}(z)=\widetilde{M}(f^{(-1)}(\omega))$,

into

$$e^{t(g(z)\partial_z+q(z))}\ \tilde{H}(z)\ e^{-t(g(z)\partial_z+q(z))}\ \widetilde{M}(z)$$

$$=\widetilde{H}(FL(z,t)\ \tilde{M}(z)=[e^{tg(z)\partial_z}\ \widetilde{H}(z)]\ \widetilde{M}(z),$$

so, removing tildes over our analytic functions and abstracting once again,

$$e^{t(g(z)\partial_z+q(z))}\ H(z)\ e^{-t(g(z)\partial_z+q(z))}\odot=H(FL(z,t))\odot$$

$$=H[f^{(-1}(f(z)+t)]\odot=[e^{tg(z)\partial_z}\ H(z)]\odot,$$

and, consistent with multiplication of conjugated ops,

$$[e^{t(g(z)\partial_z+q(z))}\ H(z)\ e^{-t(g(z)\partial_z+q(z))}]^n\odot$$

$$=e^{t(g(z)\partial_z+q(z))}\ (H(z))^n\ e^{-t(g(z)\partial_z+q(z))}\odot$$

$$=[H(FL(z,t))]^n\ \odot.$$

Switching the sign of t gives the inverse relations

$$e^{-t(g(z)\partial_z+q(z))}\ H(z)\ e^{t(g(z)\partial_z+q(z))}\odot=H(FL(z,-t))\odot=H(FL^{(-1)}(z,t))\odot$$

$$=H[f^{(-1}(f(z)-t)]\odot=[e^{-tg(z)\partial_z}\ H(z)]\ \odot.$$

Reciprocals of compositional inverses

In notation similar to Alexandrov's but letting $f_A(z,1)$ be Alexandrov's $f(z)$ while here in these notes the meaning of $f(z)$ is retained as above, consider

$$f_A(z,t)=\exp[t\ \textstyle\sum_{n\geq 1}\ \alpha_n z^{n+1}\ \partial_z]\ z=f^{(-1)}[f(z)+t]=FL(z,t)$$

with $g(x) = \frac{1}{\partial_z f(z)}$.

With

$$z = \frac{1}{u} = \frac{1}{1-(1-u)} = \sum_{n \geq 0} (1-u)^n,$$

then

$$\tilde{f}_A(z,t) = \exp[t \ \sum_{n \geq 1} \ \alpha_n z^{1-n} \ \partial_z] \ z$$

$$= \exp[-t \ \sum_{n \geq 1} \ \alpha_n u^{n+1} \ \partial_u] \ \frac{1}{u}$$

$$= \sum_{n \geq 0} (1-u)^n \ |_{u=FL(u,-t)} = \frac{1}{FL(u,-t)} = \frac{1}{FL^{(-1)}(u,t)}$$

$$= \frac{1}{FL^{(-1)}(\frac{1}{z},t)} = \frac{1}{f_A^{-1}(\frac{1}{z},t)}.$$

Cutting out the intermediate steps gives

$$\tilde{f}_A(z,t) = \frac{1}{f_A^{(-1)}(\frac{1}{z},t)},$$

which, with $t = 1$ is

$$\tilde{f}_A(z,1) = \frac{1}{f_A^{(-1)}(\frac{1}{z},1)},$$

consistent with Alexandrov's equation 1.122 on p. 21.

Note 1.108 on p. 19, for $m = 3$,

$$(z^2 \partial_z)^3 = zzDzzDzzD = z \ (zDz)^3 \ z^{-1} = z \ z^n D^n z^n \ z^{-1}$$

$$= z^n \ z \ D^n z^n \ z^{-1} = z^n Lah_n(- : zD :),$$

is conjugation of the Laguerre operator and

$$Lah_n(x) = n! \ \sum_{k=1} (-1)^n \binom{n-1}{k-1} \ \frac{x^n}{n!}$$

are the signed Lah polynomials, or normalized Laguerre polynomials of order -1, with e.g.f.

$$e^{t\ \frac{t}{t-1}} = e^{t\ Lah.(x)},$$

see [A105278] and links therein, the refined Lah polynomials are [A130561], also called the elementary Schur polynomials / functions.

Special case $q(z) = \partial_z\ g(z) = g'(z)$***:***

The conjugation

$$(g'(z) + g(z)\partial_z)^n = (\partial_z\ g(z))^n = \partial_z\ (g(z)\partial_z)^n\ \partial_z^{-1}$$

applies, so

$$e^{t(g'(z)+g(z)\partial_z)}z = e^{t\partial_z\ g(z)}z$$

$$= \partial_z\ e^{tg(z)\partial_z}\ \partial^{-1}\ z = \partial_z\ e^{tg(z)\partial_z}\ \frac{z^2}{2} = \partial_z\ \frac{(FL(z,t))^2}{2}$$

$$= FL(z,t)\ \partial_z FL(z,t).$$

Then acting on an analytic function

$$e^{t(g'(z)+g(z)\partial_z)}H(z) = e^{t\partial_z\ g(z)}H(z) = H(FL(z,t))\ \partial_z FL(z,t).$$

Let's check the consistency with the general method.

Reprising,

$$e^{t(g(z)\partial_z + q(z))}H(z) = \frac{V(f(z)+t)}{V(f(z))}\ H(f^{(-1)}(f(z)+t))$$

$$= \frac{V(f(z)+t)}{V(f(z))}\ H(FL(z,t))$$

with $v(z) = \partial_z\ \ln(V(z))$ and $q(z) = v(f(z))$,

Let $V(z) = \partial_z\ f^{(-1)}(z)$.

The inverse function theorem gives $f'(z) = \frac{1}{(f^{(-1)})'(f(z))}$, so

$$\frac{V(f(z)+t)}{V(f(z))} = \frac{(f^{(-1)})'(f(z)+t)}{(f^{(-1)})'(f(z))}$$

$$= (f^{(-1)})'(f(z)+t)f'(z) = \partial_z \; FL(z,t).$$

Also,

$$v(z) = \partial_z \ln(V(z)) = \partial_z \ln((f^{(-1)})'(z))$$

so

$$q(z) = v(f(z)) = \partial_z \ln((f^{(-1)})'(z)) \;|_{z \to f(z)}$$

$$= g(z)\partial_z \;\; \ln((f^{(-1)})'(f(z))) = g(z)\partial_z \;\; \ln(\frac{1}{f'(z)})$$

$$= g(z)\frac{-f''(z)}{f'(z)} = \frac{-f''(z)}{(f'(z))^2} = g'(z).$$

Having established that choosing $V(z) = (f^{(-1)})'(z)$ implies $q(z) = g'(z)$ and $\frac{V(f(z)+t)}{V(f(z))} = \partial_z \; FL(z,t)$, we have consistently that

$$e^{t(g'(z)+g(z)\partial_z)}H(z) = H(FL(z,t)) \; \partial_z FL(z,t).$$

An equivalent op rep is

$$e^{t(g'(z)+g(z)\partial_z)} \odot = (\partial_z FL(z,t)) \; e^{tg(z)\partial_z} \;\; \odot \, .$$

Special case $q(z) = \partial_z \; \alpha \; g(z) = \alpha \; g'(z)$:

Following the same line of argument as in the second method for solving the previous special case, let

$$V(z) = \partial_z \; (f^{(-1)}(z))^{\alpha}.$$

Then $q(z) = \alpha\ g'(z)$ and

$$e^{t(\alpha g'(z)+g(z)\partial_z)} H(z) = H(FL(z,t))\ (\partial_z FL(z,t))^{\alpha},$$

or, equivalently,

$$e^{t(\alpha g'(z)+g(z)\partial_z)}\ \odot = (\partial_z FL(z,t))^{\alpha}\ e^{tg(z)\partial_z}\ \ \odot\ .$$

This is in agreement with Alexandrov's formulas, again being careful to distinguish his f from ours by denoting his function as $f_A(z,1) = e^{g(z)\partial_z} z = FL(z,1)$ in terms of our flow function $FL(z,t) = e^{g(z)\partial_z} z = f^{(-1)}(f(z)+t)$.

Relation of solution to the finite difference op and inverse Todd operator

Now after building some confidence in the validity of the formalism, let's connect it to an alternative method of solution presented in Alexandrov related to an instance of the Baker-Campbell-Hausdorf-Dynkin theorem

$$e^{\alpha\ h(z)+g(z)\partial_z} = e^{\alpha\ n(z)} e^{g(z)\partial_z}$$

with

$$n(z) = \frac{e^{g(z)\partial_z}-1}{g(z)\partial_z}\ h(z).$$

Identifying $t = 1$ and $q(z) = \alpha\ h(z)$ in our more general formula, then

$$e^{\alpha\ n(z)} = \frac{V(f(z)+t)}{V(f(z))}\ |_{t=1},$$

and

$$n(z) = \frac{1}{\alpha}\ \ln(\frac{V(f(z)+t)}{V(f(z))})\ |_{t=1}$$

$$= \frac{1}{\alpha}\ [\ln(V(f(z)+t)) - \ln(V(f(z)))]\ |_{t=1}$$

$$= (e^{t\partial_\omega} - 1)\ \frac{1}{\alpha}\ \ln[V(\omega)]\ |_{t=1,\omega=f(z)},$$

but

$$v(z) = \partial_z \ \ln(V(z)) \text{ and } q(z) = v(f(z)) = \alpha h(z),$$

so

$$n(z) = \frac{e^{t\partial_\omega} - 1}{\partial_\omega} \ \partial_\omega \frac{1}{\alpha} \ \ln[(V(\omega))] \ |_{t=1, \omega=f(z)}$$

$$= \frac{e^{t\partial_\omega} - 1}{\partial_\omega} \ \frac{1}{\alpha} \ \ v(\omega)|_{t=1, \omega=f(z)}$$

$$= \frac{e^{tg(z)\partial_z} - 1}{g(z)\partial_z} \ \frac{1}{\alpha} \ \ v(f(z)) \ |_{t=1}$$

$$= \frac{e^{tg(z)\partial_z} - 1}{g(z)\partial_z} \ \ h(z) \ |_{t=1},$$

in agreement with Alexandrov's eqns. 1.126 and 1.127 on p. 23, for which Alexandrov invokes the BCHD theorem. With the characterization in this set of notes, we don't need to invoke this form of the BCHD, rather it drops out of our formalism. See section "5.4 The derivative of the exponential map" of the book "Lie Groups, Lie Algebras, and Representations" (second edition) by Brian Hall for more discussions on operators of this type that are used in a proof of the BCHD theorem.

Inverting, using the Todd operator,

$$h(z) = \frac{g(z)\partial_z}{e^{g(z)\partial_z} - 1} n(z)$$

$$= \sum_{n \geq 0} b_n \frac{(g(z)\partial_z)^n}{n!} n(z),$$

where b_n are the celebrated Bernoulli numbers. Starting with t small rather than $t = 1$ would introduce a convergence factor, allowing for a wider range of convergence in the Todd series and its inverse.

Generalized finite difference approach.

Reprising,

$$e^{t(g(z)\partial_z + q(z))} H(z) = \frac{V(f(z)+t)}{V(f(z))}\ H(f^{(-1)}(f(z)+t))$$

$$= \frac{V(f(z)+t)}{V(f(z))}\ H(FL(z,t))$$

with $v(z) = \partial_z\ \ln(V(z))$ and $q(z) = v(f(z)),$ so

$$e^{t(g(z)\partial_z + q(z))}\ \odot = \frac{V(f(z)+t)}{V(f(z))}\ e^{tg(z)\partial_z}\ \odot,$$

specializing to

$$e^{t(g(z)\partial_z + q(z))}\ 1 = \frac{V(f(z)+t)}{V(f(z))}$$

and

$$e^{tg(z)\partial_z}\ z = f^{(-1)}(f(z)+t) = FL(z,t).$$

Define $n(z,t)$ by

$$e^{n(z,t)} = \frac{V(f(z)+t)}{V(f(z))}.$$

Then

$$n(z,t) = \ln(\frac{V(f(z)+t)}{V(f(z))})$$

$$= [\ln(V(f(z)+t)) - \ln(V(f(z)))]$$

$$= (e^{t\partial_\omega} - 1)\ \ln[V(\omega)]\ |_{\omega=f(z)},$$

$$= \frac{e^{t\partial_\omega}-1}{t\partial_\omega}\ t\partial_\omega \ln[(V(\omega))]\ |_{\omega=f(z)}$$

$$= t\frac{e^{t\partial_\omega}-1}{t\partial_\omega}\ \ v(\omega)\ |_{t=1,\omega=f(z)}$$

$$= t\frac{e^{tg(z)\partial_z}-1}{tg(z)\partial_z}\ v(f(z))$$

$= t\frac{e^{tg(z)\partial_z}-1}{tg(z)\partial_z} \; q(z),$

and multiplying through by the inverting Bernoulli/Euler/Todd op gives, consistently,

$$q(z) = \partial_z \; \ln(V(z)) \mid_{z \to f(z)} = v(f(z))$$

our original identity for characterizing $V(z)$.

Alexandrov's special case $q(z) = -\alpha \frac{g(z)}{z^k}$:

With $q(z) = -\alpha \frac{g(z)}{z^k}$, for $k \neq 1$, otherwise k can formally be any real or complex number/variable/function/… independent of z (for $k = 1$, see further below) ,

$$n(z,t) = \frac{e^{tg(z)\partial_z}-1}{g(z)\partial_z}(-\alpha)\frac{g(z)}{z^k}$$

$$= \frac{e^{tg(z)\partial_z}-1}{g(z)\partial_z} g(z)\partial_z(\frac{\alpha}{k-1})\frac{1}{z^{k-1}}$$

$$= (e^{tg(z)\partial_z}-1)(\frac{\alpha}{k-1})\frac{1}{z^{k-1}} = \frac{\alpha}{k-1}[\frac{1}{(FL(z,t))^{k-1}} - \frac{1}{z^{k-1}}] = (e^{tg(z)\partial_z}-1)(\frac{\alpha}{k-1})\frac{1}{z^{k-1}}$$
$$= \frac{\alpha}{k-1}[\frac{1}{(FL(z,t))^{k-1}} - \frac{1}{z^{k-1}}]$$

$$= \frac{\alpha}{k-1}[\frac{1}{(FL(z,t))^{k-1}} - \frac{1}{(FL(z,0))^{k-1}}]$$

so

$$\frac{V(f(z)+t)}{V(f(z))} = e^{n(z,t)} = \frac{\exp[\frac{\alpha}{k-1}\,\frac{1}{(FL(z,t))^{k-1}}]}{\exp[\frac{\alpha}{k-1}\,\frac{1}{z^{k-1}}]}$$

$$= \frac{\exp[\frac{\alpha}{k-1}\,\frac{1}{(FL(z,t))^{k-1}}]}{\exp[\frac{\alpha}{k-1}\,\frac{1}{(FL(z,0))^{k-1}}]}$$

$$= \frac{\exp[\frac{\alpha}{k-1}\frac{1}{(f^{(-1)}(f(z)+t))^{k-1}}]}{\exp[\frac{\alpha}{k-1}\frac{1}{(f^{(-1)}(f(z)+0))^{k-1}}]},$$

implying

$$V(z) = \exp[\frac{\alpha}{k-1}\frac{1}{(f^{(-1)}(z))^{k-1}}].$$

Then consistently, from the general relations $v(z) = \partial_z \ln(V(z)) \text{ and } q(z) = v(f(z))$,

$$v(z) = \partial_z \ln(V(z)) = \partial_z \frac{\alpha}{k-1}\frac{1}{(f^{(-1)}(z))^{k-1}}$$

and

$$q(z) = v(f(z)) = \frac{\partial}{\partial f(z)}\frac{\alpha}{k-1}\frac{1}{z^{k-1}}$$

$$= \frac{\alpha}{k-1}\, g(z)\partial_z \frac{1}{z^{k-1}} = -\alpha\frac{g(z)}{z^k}.$$

Alternatively, ignoring the $n(z,t)$ approach and using only the general characterization

$$v(z) = q(f^{(-1)}(z)) = \partial_z \ln(V(z)),$$

then, with $q(z) = -\alpha\frac{g(z)}{z^k}$,

$$v(z) = q(f^{(-1)}(z)) = -\alpha\frac{g(f^{(-1)}(z))}{(f^{(-1)}(z))^k},$$

but

$g(z) = \frac{1}{f'(z)}$ and, by the inverse function theorem,

$$g(f^{(-1)}(z)) = \frac{1}{f'(f^{(-1)}(z))} = (f^{(-1)})'(z),$$

so

$$v(z) = q(f^{(-1)}(z)) = -\alpha\frac{g(f^{(-1)}(z))}{(f^{(-1)}(z))^k}$$

$$= -\alpha \frac{(f^{(-1)})'(z))}{(f^{(-1)}(z))^k} = \frac{\alpha}{k-1} \partial_z \frac{1}{(f^{(-1)}(z))^{k-1}}.$$

Then identify from the general relation $v(z) = \partial_z \ \ln(V(z))$,

$$V(z) = \exp[\frac{\alpha}{k-1} \frac{1}{(f^{(-1)}(z))^{k-1}}],$$

giving

$$\frac{V(f(z)+t)}{V(f(z))} = \frac{\exp[\frac{\alpha}{k-1} \frac{1}{(f^{(-1)}(f(z)+t))^{k-1}}]}{\exp[\frac{\alpha}{k-1} \frac{1}{(f^{(-1)}(f(z)))^{k-1}}]}$$

$$= \frac{\exp[\frac{\alpha}{k-1} \frac{1}{(f^{(-1)}(f(z)+t))^{k-1}}]}{\exp[\frac{\alpha}{k-1} \frac{1}{z^{k-1}}]}$$

$$= \frac{\exp[\frac{\alpha}{k-1} \frac{1}{(FL(z,t))^{k-1}}]}{\exp[\frac{\alpha}{k-1} \frac{1}{z^{k-1}}]}$$

$$= \frac{\exp[\frac{\alpha}{k-1} \frac{1}{(FL(z,t))^{k-1}}]}{\exp[\frac{\alpha}{k-1} \frac{1}{(FL(z,0))^{k-1}}]}$$

$$= \exp[\frac{\alpha}{k-1} [\frac{1}{(FL(z,t))^{k-1}} - \frac{1}{(FL(z,0))^{k-1}}]]$$

$$= \exp[\frac{\alpha}{k-1} [\frac{1}{(FL(z,t))^{k-1}} - \frac{1}{z^{k-1}}]]$$

$$= [\frac{\exp[\frac{(FL(z,0))^{1-k}}{1-k}]}{\exp[\frac{(FL(z,t))^{1-k}}{1-k}]}]^\alpha$$

$$= [\frac{\exp[\frac{z^{1-k}}{1-k}]}{\exp[\frac{(FL(z,t))^{1-k}}{1-k}]}]^\alpha .$$

For $k = 1$ and $q(z) = -\alpha \frac{g(z)}{z}$, take a limit treating k above as a real or complex number β:

$$\lim_{\epsilon \to 0^+} \lim_{\beta \to 1} \ \frac{1}{2} [\frac{(H(z))^{\beta+\epsilon-1}}{\beta+\epsilon-1} + \frac{(H(z))^{\beta-\epsilon-1}}{\beta-\epsilon-1}]$$

$$= \lim_{\epsilon \to 0^+} \ \frac{1}{2} [\frac{\exp[\epsilon \ln(H(z))]}{\epsilon} - \frac{\exp[-\epsilon \ln(H(z))]}{\epsilon}]$$

$= \ln(H(z)).$

Then under this limiting process, the expression

$$v(z) = \frac{\alpha}{k-1}\partial_z \frac{1}{(f^{(-1)}(z))^{k-1}}$$

remains unchanged for k any real or complex number except for $k = 1$, for which in the above limit

$$v(z) = \partial_z \ \alpha \ \ln[\frac{1}{f^{(-1)}(z)}] = \partial_z \ \alpha \ \ln[(\frac{1}{f^{(-1)}(z)})^{\alpha}],$$

and we can identify

$$V(z) = (\frac{1}{f^{(-1)}(z)})^{\alpha},$$

so

$$\frac{V(f(z)+t)}{V(f(z))} = (\frac{FL(x,0)}{FL(z,t)})^{\alpha} = (\frac{z}{FL(z,t)})^{\alpha}$$

for $q(z) = \frac{g(z)}{z}$.

This emphasizes a continuity in the general identity and is consistent with

$$n(z,t) = \frac{e^{tg(z)\partial_z}-1}{g(z)\partial_z}(-\alpha)\frac{g(z)}{z}$$

$$= \frac{e^{tg(z)\partial_z}-1}{g(z)\partial_z}g(z)\partial_z(-\alpha)\ln(z)$$

$$= (e^{tg(z)\partial_z} - 1)\ln[\frac{1}{z^{\alpha}}] = \ln[\frac{1}{(FL(z,t))^{\alpha}}] - \ln[\frac{1}{(FL(z,0))^{\alpha}}] = \alpha\ln[\frac{z}{FL(z,t)}].$$

Linearity to multiplicativity metamorphosis (LMM)

If $q(z) = q_1(z) + q_2(z)$,

then

$$q(z) = v(f(z)) = \partial)_\omega \ln(V(\omega) \mid_{\omega=f(z)} = g(z)\partial_z \ln(V(f(z))$$

$$v(z) = q(f^{(-1)}(z)) = q_1(f^{(-1)}(z)) + q_2(f^{(-1)}(z)) = v_1(z) + v_2(z)$$

$$= \partial_z \ln(V(z)) = \partial_z \ln(V_1(z)V_2(z)) \ = \partial_z \ln(V_1(z)) + \partial_z \ln(V_2(z)),$$

so linearity in $q(z)$ morphs into multiplicativity in $V(z)$.

More precisely,

if $q(z)$ is the sum of functions $q(z) = q_1(z) + q_2(z)$

and

$$e^{t(q_1(z)+g(z)\partial_z} \odot = \frac{V_1(f(z)+t)}{V_1(z)} e^{tg(z)\partial_z} \odot$$

with $v_1(z) = \partial_z \ \ln(V_1(z))$ and $q_1(z) = v_1(f(z)),$

and

$$e^{t(q_2(z)+g(z)\partial_z} \odot = \frac{V_2(f(z)+t)}{V_2(z)} e^{tg(z)\partial_z} \odot$$

with

$v_2(z) = \partial_z \ \ln(V_2(z))$ and $q_2(z) = v_2(f(z)),$

then

$$e^{t(q(z)+g(z)\partial_z} \odot = \frac{V(f(z)+t)}{V(z)} e^{tg(z)\partial_z} \odot$$

with

$$v(z) = \partial_z \ \ln(V_1(z)V_2(z)) = \partial_z \ \ln(V_1(z)) + \partial_z \ \ln(V_2(z)) = v_1(z) + v_2(z),$$

$$q(z) = q_1(z) + q_2(z) = v_1(f(z)) + v_2(f(z)),$$

and

$$\frac{V(f(z)+t)}{V(z)} = \frac{V_1(f(z)+t)V_1(f(z)+t)}{V_1(z)V_2(z)} = \frac{V_1(f(z)+t)}{V_1(z)}\frac{V_2(f(z)+t)}{V_2(z)}.$$

Each summand leads to its own independent quotient factor in the factorization of the exponential map of the diff op.

Naturally, the same conclusion is reached from the alternate method of solution:

$$n(z,t) = \frac{e^{g(z)\partial_z}}{g(z)\partial_z}\; q(z)$$

$$= \frac{e^{g(z)\partial_z}}{g(z)\partial_z}\; (q_1(z) + q_2(z))$$

$$= n_1(z) + n_2(z),$$

implying

$$e^{n(z,t)} = e^{n_1(z,t)+n_2(z,t)} = e^{n_1(z,t)}e^{n_2(z,t)}.$$

We see reflected above (ignoring multivaluedness) the continual interplay of multiplication and summation in the properties of the log and the exponential:

$\ln(a \cdot b) = \ln(a) + \ln(b)$ and $e^{a+b} = e^a e^b$ while $e^{\ln(a)+\ln(b)} = e^{\ln(ab)} = ab$ and $\ln(e^a e^b) = \ln(e^{a+b}) = a + b$.

And, just as summation of a number c with itself $n-1$ times is equivalent to the multiplication $n \cdot c$ and is generalized to $r \cdot c$ where r is any real number, $n \cdot q(z)$ translates into the quotient in the solution $(\frac{V(f(z)+t)}{V(f(z))})^n$ and $c \cdot q(z)$ into $(\frac{V(f(z)+t)}{V(f())})^c$ Of course the interpretation of raising a number to a non-integral power relies on the interplay of the log and the exponential, to wit, $c^r = \exp(r\ \ln(c))$, so naturally these creatures pop up in characterizing the action of our generalized Hermite raising op with its quotient factor.

Special case $g(z)\partial_z - \frac{1}{2}g'(z) - \frac{g(z)}{z}$:

We can apply this linearity to multiplicativity metamorphosis to the exponential mapping of the operator

$$g(z)\partial_z - \frac{1}{2}g'(z) - \frac{g(z)}{z}$$

of Eqn. 1.120 on p. 21 of Alexandrov, identified with subgroups of the Virasoro group.

We have already identified the quotient factors associated with $-\alpha\ g'(z)$ and $-\frac{g(z)}{z}$ as $(\partial_z FL(z,t))^{-\alpha}$ and $\frac{z}{FL(z,t)}$, respectively, so by the LMM

$$e^{t(g(z)\partial_z - \frac{1}{2}g'(z) - \frac{g(z)}{z})} \odot = \frac{z}{FL(z,t)}\sqrt{\partial_z FL(z,t)}\ e^{t(g(z)\partial_z} \odot,$$

or

$$e^{t(g(z)\partial_z - \frac{1}{2}g'(z) - \frac{g(z)}{z})} H(z) = \frac{z}{FL(z,t)}\sqrt{\partial_z FL(z,t)}\ H(FL(z,t)) = \tilde{H}(z,t),$$

consistent with Alexandrov's eqn. 1.134 on p. 23.

$\tilde{H}(z,t)$ is the solution to the evolution equation

$$(g(z)\partial_z - \frac{1}{2}g'(z) - \frac{g(z)}{z})\ \tilde{H}(z,t) = \partial_t \tilde{H}(z,t)$$

with $\tilde{H}(z,0) = H(z)$.

$$e^{t\ g(z)\partial_z - \frac{1}{2}g'(z) - \frac{g(z)}{z})} 1 = \frac{z}{FL(z,t)}\sqrt{\partial_z FL(z,t)}$$

and

$$e^{t\ g(z)\partial_z} z = f^{(-1)}(f(z)+t) = FL(z,t),$$

so

$$(g(z)\partial_z - \frac{1}{2}g'(z) - \frac{g(z)}{z})\ \frac{z}{FL(z,t)}\sqrt{\partial_z FL(z,t)} = \partial_t\ \frac{z}{FL(z,t)}\sqrt{\partial_z FL(z,t)}$$

with $\frac{z}{FL(z,0)}\sqrt{\partial_z FL(z,0)} = 1$

and

$g(z)\partial_z \; FL(z,t) = \partial_t \; FL(z,t)$

with $FL(z,0) = z$.

Once more, keep in mind when making comparisons that $FL(z,1) = f_A(z)$ is Alexandrov's $f(z)$ as distinguished from the $f(z)$ used in my notes here.

See also eqn, 6.9 on p. 27 of "Kadomtsev-Petviashvili hierarchy and generalized Kontsevich model" by Kharchev.

In a sense we have also been developing a generalized normal-ordering formalism, or operator disentanglement, where the Lie infinitesimal generator $g(z)\partial_z$ replaces the simple derivative. An analogous example is given in A094638, in which normal ordering of $(z^{\alpha}\partial_z)^n$ w.r.t. the Euler / state / number op $z\partial_z$ is discussed.
