Stirling Raisings

Creation Ops for the Faa di Bruno-Bell Polynomials

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The Faa di Bruno compositional partition polynomials of OEIS A036040 have been explored extensively by a slew of researchers whose names are often associated with them--some more often than others--among which are Scherk, Touchard, Steffensen,Bel, Riordan, and Sheffer. They are often called the exponential polynomials as well, but, most importantly in combinatorics, they can be considered the refined Stirling partition polynomials of the second kind since their combinatorial interpretation is a refinement of that for the Stirling numbers of the second kind.

Their e.g.f. is

$$e^{St2.(a_1,a_2,\ldots,.)\cdot t} = e^{(e^{a.t}-1)}.$$

with $(a.)^0 = a_0 = 1$. (For those unfamiliar with the umbral notation in this formula, see the last section of these notes.)

A raising / creation op R is an op that generates a polynomial/function/expression ψ_{n+1} in a sequence of such from its antecedent, i.e.,

$$R\,\psi_n = \psi_{n+1}.$$

Raising / creation ops are a core construct in the operational calculus of Sheffer polynomial sequences with each raising op paired with a complementary lowering / annihilation / destruction op, as in the ladder ops of Weyl-Heisenberg algebras and quantum theory, so they are rather important.

A generalized Lie / creation / raising op

The first raising / creation op I'll consider for the Bell polynomials is a specialization of the general creation op $\mathfrak{D} = q(z) + g(z)\partial_z$ I've explored in recent notes found in my blog posts "Dualities Between the Appell Raising Op and the Generalized Creation Op" and "A Creation Op, Scaled Flows, and Operator Algebras"; specifically,

$\mathfrak{R} = h_1(z) + \partial_z.$

This raising op is presented by Adler in a very nice paper bridging current research areas in the calculus and combinatorics, "Set partitions and integrable hierarchies".

If we let

$h(z) = \ln(A(z)),$

$h_n(z) = \partial_z^n \; h(z)$,

and

$q(z) = h_1(z) = \partial_z \; \ln(A(z)),$

then, according to my previous notes (from which this next first equality follows),

$$e^{t(h_1(z)+\partial_z)} \; 1 = \frac{A(z+t)}{A(z)} = e^{h(z+t)-h(z)} = e^{-h(z)} \; e^{t\partial_z} \; e^{h(z)}$$

$$= \exp[(e^{t\partial_z} - 1)h(x)] = e^{e^{h.(z)t} - h(z)} = e^{t \; St2.(h_1(z),...,.)}.$$

Acting on the string of equations with $\partial_{t=0}^n$ gives

$$(h_1(z) + \partial_z)^n \; 1 = \partial_{t=0}^n \; \frac{A(z+t)}{A(z)} = e^{-h(z)} \partial_z^n \; e^{h(z)} = St2_n(h_1(z), ..., h_n(z)),$$

showing that

$$St2_{n+1}(h_1(z), h_2(z), .., h_{n+1}(z)) = (h_1(z) + \partial_z) \; St2_n(h_1(z), h_2(z), .., h_n(z))$$

with $St2_0 = 1$; that is, $\mathfrak{R} = h_1(z) + \partial_z$ is a raising / creation op for one avatar of the refined Stirling partition polynomials of the second kind / Faa di Bruno / Bell polynomials..

Wolfram Alpha paste-and-check:

series in t of e^(f(x+t)-f(x))

1 + t f'(x) + 1/2 t^2 (f''(x) + f'(x)^2) + 1/6 t^3 (f^(3)(x) + f'(x)^3 + 3 f'(x) f''(x)) + 1/24 t^4 (f^(4)(x) + 3 f''(x)^2 + f'(x)^4 + 4 f^(3)(x) f'(x) + 6 f'(x)^2 f''(x)) + 1/120 t^5 (f^(5)(x) + f'(x)^5 + 10 f^(3)(x) f''(x) + 10 f^(3)(x) f'(x)^2 + 10 f'(x)^3 f''(x) + 5 f'(x) (f^(4)(x) + 3 f''(x)^2)) + O(t^6)
(Taylor series)

Check and generation of the first few polynomials:

$$(h_1(z) + \partial_z)\ 1 = h_1(z),$$

$$(h_1(z) + \partial_z)^2\ 1 = h_1^2(z) + h_2(z),$$

$$(h_1(z) + \partial_z)^3\ 1 = h_1^3(z) + 3h_1(z)h_2(z) + h_3(z),$$

$$(h_1(z) + \partial_z)^4\ 1 = h_1^4(z) + 6h_1^2(z)h_2(z) + 4h_1(z)h_3(z) + 3h_2^2(z) + h_4(z).$$

If one is not averse to monkeying around with the numerology, these connections become quickly apparent with

$$(q(z) + \partial_z)\ 1 = q(z),$$

$$(q(z) + \partial_z)\ q(z) = q^2(z) + q'(z),$$

$$(q(z) + \partial_z)\ (q^2(z) + q'(z)) = q^3(z) + 3q(z)q'(z) + q''(z),$$

$$(q(z) + \partial_z)\ (q^3(z) + 3q(z)q'(z) + q''(z))$$

$$= q^4(z) + 3q^2(z)q'(z) + q(z)q''(z) + 3q^2(z)q'(z) + 3(q'(z))^2 + 3q(z)q''(z) + q'''(z)$$

$$= q^4(z) + 6q^2(z)q'(z) + 4q(z)q''(z) + 3(q'(z))^2 + q'''(z).$$

This is an obvious first step in exploring the character of a diff op, so this relationship to the Stirling family must have been re-discovered several times by explorers familiar with or having access to tables on combinatorics allied with coarse or refined partition polynomials.

I naturally wanted to see how this characterization of a raising op for the Bell polynomials dovetailed with the formalism I presented for the specialized Lie derivative / creation op, not only as a check of my formalism but also as a way to incorporate the Bell-Stirling polynomials into the action of the more general creation op. (Like an artist, I'm obsessed with portraying the world from different perspectives.)

Reprising, the specialization gives

$$e^{t(h_1(z)+\partial_z)}\ 1 = \frac{V(z+t)}{V(z)} = e^{t\ St2.(h_1(z),h_2(z),...)},$$

so, for the generalized creation op with $g(z) = \frac{1}{\partial_z\ f(z)}$,

$$e^{t(h_1(f(z))+g(z)\partial_z)}\ 1 = e^{t(h_1(f(z))+\partial_{f(z)})}\ 1$$

$$= \frac{V(f(z)+t)}{V(f(z))} = e^{t\ St2.(h_1(f(z)),h_2(f(z)),...)}.$$

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An Appell raising operator

I showed some time ago in "Lagrange à la Lah" that the families of compositional partition polynomials, of which the Bell are perhaps the most famous, are all Appell Sheffer sequences in a distinguished indeterminate. (The general formalism of Appell operators are presented several different ways in a number of my posts, but "Dualities …" mentioned above suffices for understanding the relation of the logarithmic derivative to the Appell raising op,)

The Bell polynomials above are Appell polynomials w.r.t. to the distinguished function $h_1(z)$. One interpretation of this statement is

$$\partial_{h_1(z)}\ St2_n(h_1(z),..,h_n(z)) = n\ St2_{n-1}(h_1(z),..,h_{n-1}(z)).$$

Another is that its e.g.f. can be written in the form

$$A(t)e^{zt} = e^{a.t}e^{zt} = e^{t(a.+z)t} = e^{A.(z)t}.$$

and the e.g.f. for the Stirling polynomials may be expressed as

$$e^{t\ St2.(h_1(z),h_2(z),...)} = e^{(e^{h.(z)t}-h(z))} = e^{(e^{h.(z)t}-h_1(z)t-h(z))}\ e^{t\ h_1(z)t}.$$

The complementary raising op, from the Appell formalism, may be formed via the logarithm derivative as

$$R = \partial_{t=\frac{\partial}{\partial h_1(z)}} \;\ln[\; e^{(e^{h.(z)t}-h(z))}\;]$$

$$= \partial_t \;(e^{h.(z)t} - h(z))\;|_{t=\partial_{h_1(z)}} = h.(z)\exp[\;h.(z)\partial_{h_1(z)}\;]$$

$$= h_1(z) + h_2(z)\;\partial_{h_1(z)} + h_3(z)\;\frac{\partial^2_{h_1(z)}}{2!} + h_4(z)\;\frac{\partial^3_{h_1(z)}}{3!} + ...$$

(A side note: The lowering op relates the zeros of the n-th Appell polynomial to the stationary / critical points (max, min, inflection points) of the (n+1)-th polynomial, and, if the e.g.f. belongs to a class of entire functions, the log derivative relates the raising op to the zeros of that function. So, zeros play an important role in determining the nature of families of Appell polynomials.)

Check / illustration:

Suppressing the z variable,

$$R\;1 = (h_1 + h_2\;\partial_{h_1} + h_3\;\frac{\partial^2_{h_1}}{2!} + h_4\;\frac{\partial^3_{h_1}}{3!} + ...)\;1 = h_1,$$

$$R\;h_1 = (h_1 + h_2\;\partial_{h_1} + h_3\;\frac{\partial^2_{h_1}}{2!} + h_4\;\frac{\partial^3_{h_1}}{3!} + ...)\;h_1 = h_1^2 + h_2.$$

$$R\;(h_1^2 + h_2) = h_1^3 + h_1 h_2 + 2h_1 h_2 + h_3 = h_1^3 + +3h_1 h_2 + h_3.$$

A third raising op

Equation 5.5 on p.15 of "On the combinatorics of partition functions in AdS3/LCFT2" by Mvondo-She and Zoubos gives (morphed into the notation above) the following third raising op for the Bell polynomials;

$$h_1(z) + \sum_{n\geq 1}\; h_{n+1}(z)\;\frac{\partial}{\partial h_n(z)},$$

more in tune with ops found in relation to tau functions for the KdV and KP equations.

Note that the indeterminates of other compositional partition polynomials--the refined Lah partition polynomials (an o.g.f. version) and the cycle index polynomials of the symmetric groups (a 'log' version), a.k.a. the refined Stirling partition polynomials of the first kind--differ only by simple factorial factors from the indeterminates of the Bell polynomials (an e.g.f. version), so raising ops of similar form apply to them as well.

Initiation to umbral notation

I've used a period to flag an umbral variable / character. For umbral characters, in an ultimate expansion of an analytic expression into a series of summands for which each summand contains only the aggregated powers of the umbral character, say $(b.)^k$, the power can be lowered to the index as $(b.)^k = b_k$. For example,
$(b. + x)^n = \sum_{k=0}^{n} \binom{n}{k} b.^k \, x^{n-k} = \sum_{k=0}^{n} \binom{n}{k} b_k \, x^{n-k}$.

Accordingly, in a slight convenient generalization of this umbral maneuver,

$$e^{St2.(a_1,a_2,...,.)\cdot t} = \sum_{n\geq 0}(St2.(a_1, a_2, ..., .))^n \frac{t^n}{n!}$$

$$= \sum_{n\geq 0}(St2_n(a_1, a_2, ..., a_n))^n \frac{t^n}{n!}$$

and, introducing angled brackets to clarify the level at which umbral evaluation is to be done and with $(a.)^0 = a_0 = 1$,

$$e^{(e^{a.t}-1)} := e^{\langle(e^{a.t}-1)\rangle} = \exp[\langle \sum_{n\geq 1} \frac{(a.t)^n}{n!} \rangle]$$

$$= \exp[\langle \sum_{n\geq 1}(a.)^n \frac{t^n}{n!} \rangle] = \exp[\ \sum_{n\geq 1}\langle(a.)^n\rangle \frac{t^n}{n!}\ ]$$

$$= \exp[\ \sum_{n\geq 1} a_n \ \frac{t^n}{n!}\ ].$$

(Whew, what an effort in typing to use all the sigma gymnastics and how ugly. Umbral variables have been around since Blissard and Sylvester, i.e., around 160 years, but the math community on the whole is very conservative. It's simple, elegant, efficient, suggestive, and, therefore, powerful. So, as I sometimes say to a prospective dance partner, with my hand extended, “why not?”)

With the specialization $(a.)^n = a_n = x$ (NOT x^n), we have the e.g.f. $e^{x(e^t-1)}$ for the simple coarse Stirling polynomials of the second kind (A048993 and A008277) , but the umbral character $a.$ could represent diverse constructs: a numerical (Fibonacci) sequence

$(a_0, a_1, a_2, ...) = (2, 3, 5, 8, 13, ...)$; a generalized factorial sequence in the variable x, $(a.)^n = a_n = \frac{(x+n-1)!}{(x-1)!}$; a diff op sequence with D the derivative, $(a.)^n = a_n = \frac{(xD+n-1)!}{(xD-1)!}$; a matrix product, $(a.)^n = a_n = M^n$; etc. As in this last example, multiplicative operations of various sorts can be used as long as commutation is satisfied, but one can even introduce by definition $:AB:^n = A^n B^n$ for any two non-commuting ops A and B to express elegant meaningful identities; for example,

$$\frac{:xD:^n}{n!} = \binom{xD}{n} = \frac{St1_n(St2.(:xD:))}{n!}$$

where $St1_n(x)$ and $St2(x)$ are the Stirling polynomials of the first and second kinds, and, per the above discussion, we have the umbral composition

$$St1_n(St2.(x)) = \sum_{k=0}^{n} St1_{n,k}\ St2_k(x) = \sum_{k=0}^{n} \sum_{j=0}^{k} St1_{n,k}\ St2_{k,j}\ x^{k-j} = x^n.$$
