Dualities Between the Appell Raising Op and the Generalized Creation Op

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This set of notes is about dualities, in several reps and a special action, between the generalized creation op explored in my earlier post "The Creation Op" and the Appell creation/raising op, explored in a number of my posts.

Duality in a special action:

As discussed in several of my posts over many years, one way to define an Appell sequence of polynomials (using the umbral notation and maneuver $(b.)^n = b_n$) is through their moments a_n via

$$A_n(z) = (a. + z)^n = \sum_{k=0}^{n} \binom{n}{k} \; a_{n-k} \; z^n$$

with $a_0 = 1$. Another is through its e.g.f.

$$A(t) \; e^{zt} = e^{a.t} \; e^{zt} = e^{(a.+z)t} = e^{A.(z)t}$$

with $A(0) = 1$.

A third is with the raising op whose action is

$$R_A \;\; A_n(z) = A_{n+1}(z),$$

and, as derived in the section below on dualities in reps, is explicitly

$$R_A = z + q(\partial_z)$$

with

$$q(t) = \partial_t \ln(A(t)).$$

Then

$$R_A^n 1|_{z=0} = (z + q(\partial_z))^n \; 1|_{z=0} = A_n(z) \; |_{z=0} = (a. + 0)^n = a_n.$$

On the other hand the specialization

$$\mathfrak{D}_A = q(z) + \partial_z$$

of the generalized creation op characterized in my earlier post gives

$$\mathfrak{D}_A^n \ e^{xz} \ |_{z=0} = (q(z) + \partial_z)^n \ e^{xz} \ |_{z=0}$$

$$= \partial_{t=0}^n \ e^{t(q(z)+\partial_z)} e^{zx} \ |_{z=0} = \partial_{t=0}^n \ \frac{A(z+t)}{A(z)} \ e^{x(z+t)} \ |_{z=0}$$

$$= \partial_{t=0}^n \ A(t) \ e^{xt} = \partial_{t=0}^n \ e^{A.(x)t} = A_n(x),$$

but

$$A_n(0) = a_n = \partial_{t=0}^n \ A(t),$$

so

$$\mathfrak{D}_A^n \ 1 \ |_{z=0} = (q(z) + \partial_z)^n \ 1 \ |_{z=0} = a_n.$$

Consequently, we have ***the action duality***

$$(z + q(\partial_z))^n \ 1 \ |_{z=0} = (q(z) + \partial_z)^n \ 1 \ |_{z=0} = a_n.$$

Check:

$$(z + q(\partial_z))(z + q(\partial_z)) \ 1 \ |_{z=0} = (z^2 + q(\partial_z)z + zq(\partial_z) + q^2(\partial_z) \ 1 \ |_{z=0} = q_1 + q_0^2$$

and

$$(q(z) + \partial_z)^2 \ 1 \ |_{z=0} = (q(z) + \partial_z)(q(z) + \partial_z) \ 1 \ |_{z=0}$$

$$= (q^2(z) + q(z)\partial_z + \partial_z q(z) + \partial_z^2) \ 1 \ |_{z=0} = q_0^2 + q_1.$$

Check:

If $A(t) = e^{\frac{e^{bt}-1}{b}}$, then $q(z) = e^{bz}$, and

$$(z + e^{b\partial_z})^3 1 = (z + e^{b\partial_z})^2 (z+1) = (z + e^{b\partial_z})(z^2 + z + (z+b) + 1)$$

$$= (z + e^{b\partial_z})(z^2 + 2z + (b+1)) = (z^3 + 2z^2 + (b+1)z) + ((z+b)^2 + 2(z+b) + (b+1))$$

evaluated at $z = 0$ is $b^2 + 3b + 1$, and

$$(e^{bz} + \partial_z)^3 \ 1 = (e^{bz} + \partial_z)^2 \ e^{bz} = (e^{bz} + \partial_z) \ (e^{2bz} + be^{bz})$$

$$= (e^{3bz} + be^{2bz} + 2be^{2bz} + b^2 e^{bz}) \text{ evaluated at } z = 0 \text{ is } 1 + 3b + b^2.$$

Consistently,

$$A(t) = e^{\frac{e^{bt}-1}{b}} = 1 + t + (b+1)\tfrac{t^2}{2} + (b^2 + 3b + 1)\tfrac{t^3}{3!} + (b^3 + 7b^2 + 6b + 1)\tfrac{t^4}{4!} + \dots .$$

(Some will immediately recognize the coefficient polynomials as a version of the Stirling polynomials of the second kind, a.k.a. the exponential/ Scherk / Bell / Touchard / Steffensen polynomials.)

Dualities in representations

Some equivalent reps for the raising op for an Appell sequence (reprising several earlier sets of notes)

$$z + q(\partial_z) = A(\partial_z) \ z \ \tfrac{1}{A(\partial)} = z + [\ln(A(\partial_z)), z] = z + \partial_{\partial_z} \ \ln(A(\partial_z)),$$

using the commutator for two ops $[F, G] = FG - GF$.

This last expression in the string of equalities is perhaps more quickly digested, on first encounter, written as $z + \partial_{\partial_z} \ \ln(A(\partial_z)) = z + \partial_t \ \ln(A(t)) \ |_{t=\partial_z}$.

The counterpart string for the dual op is

$$\partial_z + q(z) = \tfrac{1}{A(z)} \ \partial_z \ A(z) = \partial_z + [\partial_z, \ln(A(z))] = \partial_z + \partial_z \ \ln(A(z)).$$

The second expression, a conjugation of the raising op z of the powers z^n, is perhaps the easiest way to understand the raising op for an Appell sequence. Using a generalized shift op,

$$A(\partial_z)\ z^n = e^{a.\partial_z}\ z^n = (a. + z)^n = A_n(z),$$

so, conversely,

$$\frac{1}{A(\partial_z)}\ A_n(z) = z^n.$$

It follows that

$$A(\partial_z)\ z\ \frac{1}{A(\partial_z)}\ A_n(z) = A(\partial_z)\ z^{n+1} = A_{n+1}(z),$$

a result that defines the action of the raising op R for any Sheffer sequence:
$R\ S_n(z) = S_{n+1}(z)$.

I've also made use of the Graves-Pincherle dual op derivatives

$$[f(L), R] = \frac{\partial}{\partial L}\ f(L) = f'(L)$$

and

$$[f(R), L] = \frac{\partial}{\partial R}\ f(R) = f'(R)$$

for any pair of ladder ops--a raising / creation op R and a lowering / annihilation / destruction op L, such as z and ∂_z for the power polynomials $P_n(z) = z^n$--satisfying $[L, R] = 1$. (I've derived these results multiple ways in previous sets of notes on the Appell raising op and the Pincherle derivative.)

Summarizing, we have ***the representation dualities***

$$R_A = z + q(\partial_z) = A(\partial_z)\ z\ \frac{1}{A(\partial)} = z + [\ln(A(\partial_z)), z] = z + \partial_{\partial_z}\ \ln(A(\partial_z)),$$

and

$$\mathfrak{D}_A = \partial_z + q(z) = \frac{1}{A(z)}\ \partial_z\ A(z) = \partial_z + [\partial_z, \ln(A(z))] = \partial_z + \partial_z\ \ln(A(z))$$

between the Appell raising op and a specialization of the generalized creation op.