Ruling the inverse universe,
the inviscid Hopf-Burgers evolution equation:
Compositional inversion, free probability, associahedra, diff ids, integrable hierarchies, and translation

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Introduction / Orientation (the meat of the math begins with the next section)

A mapping of associations among the iconic inviscid (Poisson-Riemann-Bateman) Hopf-Burgers nonlinear evolution / transport partial differential equation; compositional inversion; Lagrange inversion formulas; differential identities / integrable hierarchies / conservation laws; inversion partition polynomials, in particular the refined Euler characteristic partition polynomials of the associahedra; the formal free moment partition polynomials generating the free cumulants of free probability; and shifts / translations via variables and free moments is presented in these notes.

The main initial impetus in exploring the above relationships was to prove the particularly interesting novel empirical observation that the partial derivative w.r.t. the distinguished moment m_1 of the free moment partition polynomials (OEIS A350499, still in the draft queue at this time), which give the formal free cumulants in terms of the formal free moments of free probability, gives the refined Euler characteristic partition polynomials of the associahedra (A133437 and A111785) with a simple numerical factor. (See also my MathOverflow question "Combinatorics for the action of Virasoro / Kac–Schwarz operators: partition polynomials of free probability theory".) This is an analog of a result in classic probability--the partial derivative w.r.t. the moment m_1 of the classic moment partition polynomials (A127671) gives the refined Euler characteristic partition polynomials of the permutahedra (A133314, see also A049019) with a simple numerical factor. Soon it became apparent that derivatives w.r.t. all the free moments, or indeterminates of the refined Euler characteristic partition polynomials of the associahedra, are related to inviscid Hopf-Burgers nonlinear evolution p.d.e.s.

There have been a slew of papers written on nonlinear transport equations--the inviscid Hopf-Burgers nonlinear partial differential equation (IBHE) playing a key role--since at least the times of Poisson and Riemann. The applications span fields from engineering and mathematical physics to pure mathematics; hydrodynamics, traffic control, Euler-Lagrange equations, enumerative combinatorics, convex polytopes, free probability, 2-D quantum theory, string theory, and random matrices, to name a few.

The main body of these notes and the section Related Stuff at the end of this pdf ref some articles that can serve as links to the larger corpus of literature on the IBHE. Olver lists Poisson and Riemann as two of the earlier researchers (the rationale behind the use of the phrases ‘conservation laws’ and ‘integrable hierarchies’ in the context of nonlinear transport equations is explained very nicely by Olver). Dubrovin and Yang discuss the related Riemann infinite integrable hierarchy and some relations between the IBHE and other classic nonlinear p.d.e.s.. Bonkile et al. give a short history of applications in hydro- and aerodynamics. Other authors discuss in detail the complex Burgers equation and its association with Euler-Lagrange equations and limit curves. The list is potentially endless.

Other than Buchstaber and Panov, the role of the IBHE in the enumerative combinatorics of the associahedra has been relatively neglected, and even they (at least initially when I first encountered their notes) dealt only with the coarse face polynomials of the associahedra rather than the refined face partition polynomials, or refined Euler characteristic partition polynomials of the associahedra. The roles of the IBHE, compositional inversion, and the combinatorics of noncrossing partitions have been noted since Voiculescu’s early original work, or soon after, yet the relation to associahedra apparently has not encompassed in this context till now. As far as I know, Loday was the first to note the relation between associahedra and compositional inversion but not to the formal free cumulants of free probability and the IBHE. (Newton derived at least the first few partition polynomials, but didn’t construct the associahedra). The study of integrable hierarchies / conservation laws of nonlinear transport equations has a long history and continues to be a source of revelations in modern research, yet aside from B & P and a fairly recent article by Adler, partition polynomials are rarely explicitly identified in this context. Investigations of scattering amplitudes in certain quantum field theories have recently focussed on the roles of Grassmannians, noncrossing partitions, and the associahedra, yet these have not explicitly drawn the connections to the IBHE and most often even neglect the explicit role of Lagrange inversion. The utility of combinatorial / compositional Hopf algebras (e.g., the Faa di Bruno HA) with their antipodes often representing inversion of one sort or another have been a modern line of inquiry in quantum field theory, yet again the wider associations have not been fully expressed. This set of notes addresses some of these shortcomings, mapping relationships among compositional inversion, Lagrange inversion formulas, the IBHE, integrable hierarchies / conservation laws, and partition polynomials associated to noncrossing partitions and the associahedra--as such, it is an extension of my earlier notes on this constellation of topics.

Free cumulant and moment partition polynomials, associahedra, and diff ids

Compositional inversion is a major component in the definition of formal free cumulants in terms of formal free moments in the theory of free probability, related to operator algebras, random matrices and certain quantum field theories (see refs below). Given the ordinary generating function (o.g.f), or formal power series, for the formal free moments, regarded as arbitrary, i.e., bereft of any probability interpretation, possibly a divergent series,

$$M(x) = x + m_1 x^2 + m_2 x^3 + \cdots ,$$

the formal cumulants can be defined as

$$C(x) = \frac{x}{M^{(-1)}(x)} = 1 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots ,$$

where $M^{(-1)}(x)$ is the formal compositional inverse of the o.g.f. $M(x)$ defined by your preferred version of the Lagrange inversion formula (see OEIS A145271 for details on several avatars of the LIF and links to flow functions and Lie derivatives).

The first few free moment partition polynomials, giving the free cumulants in terms of the free moments (cf. A350499), are

$$c_1 = m_1$$
$$c_2 = m_2 - m_1^2$$
$$c_3 = m_3 - 3\, m_2\, m_1 + 2\, m_1^3$$
$$c_4 = m_4 - 2\, m_2^2 - 4\, m_3\, m_1 + 10\, m_2\, m_1^2 - 5\, m_1^4$$
$$c_5 = m_5 - 5\, m_2\, m_3 + 15\, m_2^2\, m_1 - 5\, m_4\, m_1 + 15\, m_3\, m_1^2 - 35\, m_2\, m_1^3 + 14\, m_1^5$$

Conversely, the first few free cumulant partition polynomials, giving the free moments in terms of the free cumulants, are (cf. A134264)

$$m_1 = c_1$$
$$m_2 = c_2 + c_1^2$$
$$m_3 = c_3 + 3\, c_1\, c_2 + c_1^3$$
$$m_4 = c_4 + 2\, c_2^2 + 4\, c_1\, c_3 + 6\, c_1^2\, c_2 + c_1^4$$
$$m_5 = c_5 + 5\, c_2\, c_3 + 5 c_1\, c_4 + 10\, c_1\, c_2^2 + 10\, c_1^2\, c_3 + 10\, c_1^3\, c_2 + c_1^5 \; .$$

Long ago, in 2012, I commented in OEIS A127671, giving the classic moment partition polynomials for the classic formal cumulants expressed in terms of the classic formal moments, that the derivative with respective to the classic moment m_1 of those moment partition polynomials generates polynomials proportional to those of A133314, the refined Euler characteristic partition polynomials of the permutahedra, regulating the combinatorics of multiplicative inversion. Naturally, I recently took the derivative of the free moment partition polynomials with respect to the free moment m_1 and noticed the first few refined Euler characteristic partition polynomials of the associahedra (OEIS A133437 and A111785) popped up. These partition polynomials regulate the compositional inversion of formal o.g.f.s, or power series. For examples of the relation between the geometrically distinct faces of the associahedra and the partition polynomials, see, e.g., the MathOverflow question "Why do polytopes pop up in Lagrange inversion?", the answer to the MO-Q "Why is there a connection between enumerative geometry and nonlinear waves?", the MO-Q "Loday's characterization and enumeration of faces of associahedra (Stasheff polytopes)", or my answer "The combinatorics underlying iterated derivatives (infinitesimal Lie generators) for compositional inversion and flow maps for vector fields" to an MO-Q on important formulas in combinatorics.

More precisely, the compositional inverse $M^{(-1)}(x)$, containing the first few inversion partition polynomials, for $M(x)$ (cf. A133437) is

$$M^{(-1)}(x) = x + (-m_1)\ x^2 + (2\ m_1^2 - m_2)\ x^3 + (-5\ m_1^3 + 5\ m_1\ m_2 - m_3)\ x^4$$

$$+(14\ m_1^4 - 21\ m_1^2\ m_2 + 6\ m_1\ m_3 + 3\ m_2^2 - m_4)\ x^5 + \cdots$$

$$= \sum_{n\geq 1} P_n\ x^n.$$

Then, e.g., taking the partial derivative w.r.t. The indeterminate m_1,

$$\partial_{m_n}\ c_5 = \partial_{m_n}\ [m_5 - 5\ m_2\ m_3 + 15\ m_2^2\ m_1 - 5\ m_4\ m_1 + 15\ m_3\ m_1^2 - 35\ m_2\ m_1^3 + 14\ m_1^5]$$

$$= 15\ m_2^2 - 5\ m_4 + 30\ m_3\ m_1 - 105\ m_2\ m_1^2 + 70\ m_1^4$$

$$= 5\ [3\ m_2^2 -\ m_4 + 6\ m_3\ m_1 - 21\ m_2\ m_1^2 + 14\ m_1^4$$

$$= 5\ P_5.$$

Motivated by this empirical evidence, I want to show the formal validity of the partial differential identity (diff id)

$$\partial_{m_1}\ C(x) = x\partial_x\ M^{(-1)}(x),$$

which is equivalent to

$$\partial_{m_1}\ \frac{C(x)}{x} = \partial_{m_1}\ \frac{1}{M^{(-1)}(x)} = \partial_x\ M^{(-1)}(x) = \partial_x\ \frac{x}{C(x)},$$

which, in turn, is equivalent to

$$\partial_{m_1}\ M^{(-1)}(x) = -\partial_x\ \frac{(M^{(-1)}(x))^3}{3}.$$

which suggests the more general identity

$$\partial_{m_n} \; M^{(-1)}(x) = -\partial_x \; \frac{(M^{(-1)}(x))^{n+2}}{n+2}.$$

I'll show the validity of these identities and that these diff ids, an avatar of the Lagrange inversion formula, and associated inviscid Hopf-Burgers nonlinear evolution p.d.e.s can be encoded in the general formal identity

$$e^{\alpha \; \partial_{m_n}} \; H(x; m_1, m_2 .., m_n, ..) = H(x; m_1, m_2 .., m_n + \alpha, ..)$$

$$= \sum_{k \geq 0} (-\alpha)^k \; \frac{\partial_x^k}{k!} \; \frac{H(x; m_1, .., m_n, ..)^{k+1}}{k+1}$$

with

$$H(x; m_1, m_2 .., m_n, ..) = (M^{(-1)}(x; m_1, .., m_n, ..))^{n+1}$$

for $n \geq 1$.

This form with $n = 0$ encompasses the iconic IBHE if we include as presented explicitly in later sections a 'time' variable as $t = m_0$ and

$$H(x, t; m_1, m_2, .., m_n, ..) = H(x, m_0; m_1, m_2, .., m_n, ..).$$

Then to first order in α,

$$\partial_t \; M^{(-1)}(x, t; m_1, .., m_n, ..) = \partial_{m_0} \frac{(M^{(-1)}(x, m_0; m_1, .., m_n, ..))^2}{2}$$
$$= -\partial_x \; \frac{(M^{(-1)}(x, m_0; m_1, .., m_n, ..))^2}{2} = -\partial_x \; \frac{(M^{(-1)}(x, t; m_1, .., m_n, ..))^2}{2}.$$

Laplace transform and compositional inversion

To formally generate this hierarchy of identities, I'll make use of a formal relation between compositional inversion and the Laplace transform I presented in Lagrange à la Lah that leads to the classic Lagrange inversion formula. Later I'll use the inviscid Hopf-Burgers nonlinear evolution partial differential equation (p.d.e.) to validate these identities.

Via the change of variable $zu = f^{(-1)}(z\nu)$ and interchange of operations, for suitable analytic functions,

$$\int_0^\infty \exp[-\frac{f(zu)}{z}\ ]\ du = \int_0^\infty e^{zu\ \partial_{\omega=0}}\ \exp[-u\frac{f(\omega)}{\omega}\ ]\ du$$

$$= \sum_{n\geq 0} z^n \partial^n_{\omega=0} \int_0^\infty \frac{u^n}{n!} \exp[-u\frac{f(\omega)}{\omega}\ ]du = \sum_{n\geq 0}\ \partial^n_{\omega=0}\ [\frac{\omega}{f(\omega)}]^{n+1}\ z^n$$

$$= \int_0^\infty\ \frac{1}{z}\ \exp[-\nu\frac{1}{z}\ ]\ \partial_\nu\ f^{(-1)}(\nu)\ d\nu$$

$$= \sum_{n\geq 0}\ \partial^n_{\nu=0}(f^{(-1)})'(\nu)\ z^n = \sum_{n\geq 0}\ \partial^{n+1}_{\nu=0} f^{(-1)}(\nu)\ z^n.$$

The last Laplace transform, the classic Borel-Laplace transform, useful in summation of divergent series, is effected from the first integral through the simple change of variable for a suitable compositional inverse pair of functions / power series $y = f(x)$ and $y = f^{(-1)}(x)$. Although the class of functions satisfying the above transform identity is restrictive, the final classic Lagrange inversion formula

$$\partial^{n+1}_{\nu=0} f^{(-1)}(\nu) = \partial^n_{\omega=0}\ [\frac{\omega}{f(\omega)}]^{n+1}$$

connecting the coefficients of a formal Taylor series, or exponential generating function (e.g.f.), with those of the inverse formal Taylor series is valid for convergent series and, with respect to the two sets of formal Taylor series coefficients, is an involution that can be taken as the definition of a compositional inverse pair of divergent series.

Remarks: In some sense, this is analogous to the reverse of Euler summation of a divergent geometric series and represents a generalized infinite-dimensional analytic continuation of the Laplace transform in the indeterminates--the coefficients of a series denoted $f(x)$ about the origin--from those for which the Borel summation is valid to those for which it isn't. In addition,, in Lagrange a la Lah, I show how various families of Lagrange inversion partition polynomials can be generated by expanding the exponential in the first integral using different families of composition partition polynomials (the e.g.f. rep, i.e, the Faa di Bruno / refined Stirling polynomials of the second kind; the o.g.f. rep, i.e, the refined Lah polynomials; a log rep;the shifted reciprocal rep; and the Stirling polynomials of the first kind) and then taking the resulting Laplace transform formally, i.e., term by term. These correspond to the same partition polynomials obtained by the classic Lagrange inversion formula or via iterated Lie derivatives.

Cauchy contour integration and compositional inversion

One of several ways to confirm that the compositional inversion formula derived from the Laplace transform maneuvers above,

$$\partial_{\nu=0}^{n+1} f^{(-1)}(\nu) = \partial_{\omega=0}^{n} \, [\tfrac{\omega}{f(\omega)}]^{n+1},$$

Is valid while also showing that invoking an infinite-dimensional analytic continuation is valid in this case is through the local Cauchy contour transform for functions analytic in a disk about z in the complex plane

$$F(z) = \tfrac{1}{2\pi i} \oint \tfrac{F(\omega)}{\omega - z} \, d\omega$$

for a closed contour about z within the disk. Differentiating w.r.t. z gives a contour integral rep for the derivative

$$\partial_z^n \, F(z) = \tfrac{n!}{2\pi i} \oint \tfrac{F(\omega)}{(\omega - z)^{n+1}} \, d\omega,$$

and, in agreement with the Lagrange inversion formula above,

$$\partial_{z=0}^{n} \, (\tfrac{z}{f(z)})^{n+1} = \tfrac{n!}{2\pi i} \oint \tfrac{(\frac{\omega}{f(\omega)})^{n+1}}{\omega^{n+1}} \, d\omega$$

$$= \tfrac{n!}{2\pi i} \oint \tfrac{1}{f(\omega)^{n+1}} \, d\omega = \tfrac{n!}{2\pi i} \oint \tfrac{(f^{(-1)})'(\alpha)}{\alpha^{n+1}} \, d\alpha$$

$$= \partial_{\alpha=0}^{n} \, (f^{(-1)})'(\alpha) = \partial_{z=0}^{n+1} \, f^{(-1)}(z)$$

with the change of variable $\omega = f^{(-1)}(\alpha)$ for a contour about $\alpha = 0$.

Taylor series, translation, Lie derivatives and compositional inversion

Other proofs that the associahedra family of inversion partition polynomials are equivalent to Lagrange inversion involve an iterated Lie derivative or, equivalently, simply transforming the Taylor series for an analytic.

For a function analytic at u and sufficiently small $|y - x|$, the translation property

$e^{(y-x)\partial_u} \, f(u) = f(u + y - x)$ holds.

Then the Taylor series about y is

$e^{(y-x)\partial_u}\ f(u)\ |_{u=x} = f(y-x+x) = f(y) = \sum_{n\ 0}\ f^{(n)}(x)\ \frac{(y-x)^n}{n!}.$

For $x = 0$ and $y = f^{(-1)}(\omega)$,

$$f(f^{(-1)}(\omega)) = \omega\ = \sum_{n\geq 0}\ [(\frac{\partial}{\partial y})^n]\ f(y)\ |_{y=0})]\frac{(f^{(-1)}(\omega))^n}{n!}$$

$$= \sum_{n\geq 0}\ [(\frac{1}{(f^{(-1)})'(u)}\ \frac{\partial}{\partial u})^n\ u|_{u=0}]\frac{(f^{(-1)}(\omega))^n}{n!},$$

so

$$[(\frac{1}{(f^{(-1)})'(u)}\ \frac{\partial}{\partial u})^n\ u\ |_{u=0})]$$

must be the Taylor series coefficients about the origin for the formal Taylor series represented by $f(y)$.

Alternatively, with $y = f^{(-1)}(x)$ and $g(x) = \frac{1}{(f^{(-1)})'(x)}$,

$$e^{z\ g(x)\partial_x}\ x = e^{z\ \partial_y}\ f(y) = f(y+z) = f[f^{(-1)}(x)+z].$$

Then for $f(0) = 0 = f^{(-1)}(0)$ and $(f^{(-1)})'(0) \neq 0$,

$$f[f^{(-1)}(x)+z]\ |_{x=0} = f(z) = e^{z\ g(x)\partial_x}\ x\ |_{x=0},$$

so, again,

$$(g(x)\partial_x)^n\ x\ |_{x=0} = (\frac{1}{(f^{(-1)})'(x)}\frac{\partial}{\partial x})^n\ x\ |_{x=0} = f^{(n)}(0)$$

gives the Taylor series coefficients of $f(z)$. All roads lead to Rome ... or rather Berlin/Paris (Lagrange) ... and London (Newton).

Note if $f(0) \neq 0$, let $\tilde{f}(x) = f(x) - f(0)$, then $f^{(-1)}(x) = \tilde{f}^{(-1)}(x - f(0))$ is the local inverse.

One formulation may be more useful and intuitive than the others in a given context, and, of course, there are the purely algebraic and combinatorial ones. The last two derivations show how shifts in/translation of a variable and compositional inversion are related. These two operations are also connected via the inviscid Burgers-Hopf equation to another avatar of Lagrange inversion below.

Laplace transform approach and derivatives w.r.t. indeterminates

Returning to the original Laplace transform relation, with $f(z) = M^{(-1)}(z)$, then $\partial_{\nu=0}^{n+1} \; M(\nu) = (n+1)! \; m_n$ and the transform relation becomes

$$\sum_{n \geq 0} (n+1)! \; m_n \; z^n = \int_0^\infty \; \exp[-\frac{M^{(-1)}(zu)}{z} \;] \; du.$$

Taking the partial derivative w.r.t. m_1 of both sides gives

$$2z = \int_0^\infty \; \partial_{m_1} \exp[-\frac{M^{(-1)}(zu)}{z} \;] \; du$$

$$= \int_0^\infty \; [-\partial_{m_1} \; \frac{M^{(-1)}(zu)}{z} \;] \; \exp[-\frac{M^{(-1)}(zu)}{z} \;] \; du.$$

Assuming the validity of the identity

$$\partial_{m_1} \; M^{(-1)}(x) = -\partial_x \; \frac{(M^{(-1)}(x))^3}{3} = -(M^{(-1)}(x))^2 \; \partial_x \; M^{(-1)}(x),$$

then

$$-\partial_{m_1} \; \frac{M^{(-1)}(zu)}{z} \; = \frac{1}{z} \; \partial_{zu} \; \frac{(M^{(-1)}(zu))^3}{3} = \frac{1}{z} \; (M^{(-1)}(zu))^2 \; \partial_{zu} M^{(-1)}(zu),$$

and

$$2z = \int_0^\infty \; \partial_{m_1} \exp[-\frac{M^{(-1)}(zu)}{z} \;] \; du$$

$$= \int_0^\infty \; [-\partial_{m_1} \; \frac{M^{(-1)}(zu)}{z} \;] \; \exp[-\frac{M^{(-1)}(zu)}{z} \;] \; du$$

$$= \int_0^\infty \; \exp[-\frac{M^{(-1)}(zu)}{z} \;] \; \frac{1}{z^2} \; (M^{(-1)}(zu))^2 \; \partial_{zu} M^{(-1)}(zu) \; d(zu)$$

$$= \frac{1}{z^2} \int_0^\infty \; \exp[-\frac{1}{z}\beta] \; \beta^2 \; d\beta = \frac{1}{z^2} \; 2z^3 = 2z,$$

where $\beta = M^{(-1)}(zu)$.

Since the results of the Laplace transform are unique,

$$\partial_{m_1} \; M^{(-1)}(z) = -\partial_z \; \frac{(M^{(-1)}(z))^3}{3} = -(M^{(-1)}(z))^2 \; \partial_z M^{(-1)}(z).$$

Following the same line of argument, it follows that

$$\partial_{m_n} \; M^{(-1)}(z) = -\partial_z \; \frac{(M^{(-1)}(z))^{n+2}}{n+2} = -(M^{(-1)}(z))^{n+1} \; \partial_z M^{(-1)}(z).$$

Couched in terms of the formal cumulants,

$$\partial_{m_n} \; \frac{z}{C(z)} = -\partial_z \; \frac{(\frac{z}{C(z)})^{n+2}}{n+2}.$$

Again the transformation of the integrals are only strictly valid for series whose coefficients have restrictions, but once again I invoke a generalized infinite-dimensional 'analytic continuation' to indeterminates / coefficients outside this restricted range and claim the diff ids apply to any arbitrary pair of series formally inverse to each other via the Lagrange inversion partition polynomials, i.e., whose sets of coefficients are related via the inversion involution. This is validated via the ICBH in arguments below.

This establishes a differential means of evaluating two types of self-convolutions of the families of Lagrange inversion partition polynomials since binomial convolutions of the coefficients of e.g.f.s correspond to multiplication of their e.g.f.s and Cauchy convolution, to o.g.f.s. A while ago, I used the inviscid Burgers-Hopf partial differential equation (IBHE) to determine a single convolution. The formula above provides a simple method to determine the higher order self-convolutions.

The inviscid Hopf-Burgers equation and compositional inversion

In "Quantum Deformation Theory of the Airy Curve and Mirror Symmetry of a Point", Zhou states that if a function $u(x,t)$ satisfies

$$\partial_t \; u = \partial_x \; \frac{u^{n+1}}{(n+1)!}$$

for $n > 1$, then, $\nu = \frac{u^n}{n!}$ satisfies the IBHE

$\partial_t \ \nu = \partial_x \ \frac{\nu^2}{2} = \nu \ \partial_x \ \nu.$

This follows simply from

$$\partial_t \ \nu = \partial_t \ \frac{u^n}{n!} = \frac{u^{n-1}}{(n-1)!} \ \partial_t \ u = \ \frac{u^{n-1}}{(n-1)!} \ \partial_x \ \frac{u^{n+1}}{(n+1)!}$$

$$= \frac{u^{n-1}}{(n-1)!} \ \frac{u^n}{n!} \ \partial_x \ u = \frac{u^n}{n!} \ \partial_x \ \frac{u^n}{n!} = \nu \ \partial_x \ \nu.$$

Then we can morph each identity of our infinite set of differential identities

$$\partial_{m_n} \ M^{(-1)}(x) = -\partial_x \ \frac{(M^{(-1)}(x))^{n+2}}{n+2}$$

into

$$\partial_{t_p} \ M^{(-1)}(x) = \partial_x \ \frac{(M^{(-1)}(x))^{p+1}}{(p+1)!}$$

with $t_p = -(n+1)! \ m_n \text{ and } p = n+1,$

and then into the IBHE

$$\partial_{t_p} \ \nu = \partial_x \ \frac{\nu^2}{2} = \nu \ \partial_x \ \nu$$

with $\nu = \frac{(M^{(-1)}(x))^p}{p!} = \frac{(M^{(-1)}(x))^{n+1}}{(n+1)!}$.

In “Toric Topology”, Buchstaber and Panov note (dovetailing their notation with mine) that the infinite hierarchy of 'conservation laws'

$ $\partial_{t_p} \frac{\nu^k}{k} = \partial_x \frac{\nu^{k+1}}{k+1}$

hold for solutions to an IBHE for $k \geq 1$ since

$$\partial_{t_p} \frac{\nu^k}{k} = \nu^{k-1} \ \ \partial_{t_p} \ \nu \text{ and } \partial_x \frac{\nu^{k+1}}{k+1} = \nu^k \ \partial_x \ \nu.$$

Then

$$\partial_{t_p} \frac{(M^{(-1)}(x))^{kp}}{k \ (p!)^k} = \ \partial_x \frac{(M^{(-1)}(x))^{(k+1)p}}{(k+1) \ (p!)^{k+1}},$$

and

$$\partial_{m_n} \frac{(M^{(-1)}(x))^{k(n+1)}}{k} = -\partial_x \frac{(M^{(-1)}(x))^{(k+1)(n+1)}}{(k+1)}.$$

Check:

For $n \geq 1$,

$$\partial_{m_n} \ M^{(-1)}(x) = -\partial_x \ \frac{(M^{(-1)}(x))^{n+2}}{n+2} = -(M^{(-1)}(x))^{n+1} \ \partial_x \ M^{(-1)}(x),$$

so

$$\partial_{m_n} \frac{(M^{(-1)}(x))^{k(n+1)}}{k} = (n+1) \ (M^{(-1)}(x))^{[k(n+1)-1]} \ \partial_{m_1} \ M^{(-1)}(x)$$

$$= -(n+1) \ (M^{(-1)}(x))^{(n+1)(k+1)-1} \ \partial_x \ M^{(-1)}(x) = -\partial_x \frac{(M^{(-1)}(x))^{(n+1)(k+1)}}{(k+1)}.$$

From my notes "Compositional Inverse Pairs, the Inviscid Burgers-Hopf Equation, and the Stasheff Associahedra", a solution of the IBHE

$$\partial_t \bar{U}(x,t) = -\partial_x \ \frac{\bar{U}^2(x,t)}{2}$$

(note the sign) can be constructed from

$$M(x,t) = x + t \cdot (m_1 x^2 + m_2 x^3 + m_4 x^5 + \cdots) = x + t \ F(x)$$

and its compositional inverse in x about the origin

$$M^{(-1)}(x,t) = x - m_1 t x^2 + (2m_1^2 t^2 - m_2 t) x^3 + (-5m_1^3 t^3 + 5m_1 m_2 t^2 - m_3 t) x^4$$

$$+(14m_1^4 t^4 - 21m_1^2 m_2 t^3 + (6m_1 m_3 + 3m_2^2) t^2 - m_4 t) x^5 + \cdots$$

as

$$\bar{U}(x,t) = \frac{x - M^{(-1)}(x,t)}{t}$$

with $\bar{U}(x,0) = F(x) = \frac{M(x,t)-x}{t}$ and $\bar{U}(x,1) = x - M^{(-1)}(x,1)$.

This IBHE can be regarded as the p.d.e. governing the deformation of the curve $y = F(x) = \bar{U}(x,0)$ into the curve $y = \bar{U}(x,1) = x - M^{(-1)}(x,1)$ over unit time via the continuous transformation $y = \bar{U}(x,t)$. This is easily visualized for the special case $m_1 = 1$ and $m_n = 0$ otherwise, giving generators for the Catalan numbers, with

$$M(x,t) = x + tx^2 = x(1+tx),$$

$$\bar{U}_L(x,t) = \frac{x - \frac{-1-\sqrt{1+4xt}}{2t}}{t} = 1/t^2 + (2x)/t - x^2 + 2tx^3 - 5t^2x^4 + 14t^3x^5 + \cdots,$$

$$\bar{U}_R(x,t) = \frac{x - \frac{-1+\sqrt{1+4xt}}{2t}}{t} = x^2 - 2tx^3 + 5t^2x^4 - 14t^3x^5 + 42t^4x^6 - 132t^5x^7 + \cdots,$$

and

$$y = \bar{U}(x,0) = F(x) = x^2,$$

where $y = \bar{U}_L(x,t)$ and $y = \bar{U}_R(x,t)$ together cover the full curve $\bar{U}(x,t)$. Plotting $y = \bar{U}_L(x,t)$, $y = \bar{U}_R(x,t)$, and $y = \bar{U}(x,0) = x^2$ using Desmos graphing on the Net with the value of t controlled with a slider provides a visualization of the continuous deformation. This interpretation of the IBHE as a p.d.e. governing the evolution over unit time of one curve into another is put to good use in "On the Large N Limit of the Itzykson-Zuber Integral" by Matytsin (see, in particular, eqns. 1.5, 1.6, 2.13-2.16, 3.1, and 4.4-4.8).

Since $\bar{U}(x,t)$ satisfies $\partial_t \bar{U}(x,t) = -\partial_x \frac{\bar{U}^2(x,t)}{2}$, both $U_1(x,t) = \bar{U}(-x,t)$ and $U_2(x,t) = \bar{U}(x,-t)$ satisfy $\partial_t U(x,t) = \partial_x \frac{U^2(x,t)}{2}$.

Define, for use below, a g.f. for the Catalan numbers A000108

$$Cat(x,t) = \bar{U}_R(x,-t) = -\frac{x + \frac{-1+\sqrt{1-4xt}}{2t}}{t}$$

$$= x^2 + 2tx^3 + 5t^2x^4 + 14t^3x^5 + 42t^4x^6 + 132t^5x^7 + \cdots.$$

Note the associated free cumulants for this case become the Catalan numbers also.

The IBHE provides a means of evaluating a self-convolution of the inversion partition polynomials through first order differentiation rather than multiplication of series. The results are presented in the OEIS. The integrable hierarchy

$$\partial_{m_n} \ M^{(-1)}(x,t) = -\partial_x \ \frac{(M^{(-1)}(x,t)^{n+2}}{n+2}$$

allows numerical and analytic evaluations of higher order self-convolutions for any family of inversion partition polynomials.

B & P introduce the clever iterative maneuver

$$\partial_t^k \frac{U}{1} = \partial_t^{k-1} \ \partial_x \frac{U^2}{2} = \ \partial_t^{k-2} \ \partial_x^3 \frac{U^3}{3} = \cdots = \partial_x^k \ \frac{U^{k+1}}{k+1},$$

for $k \geq 1$, which follows from the diff id we already established, for $n \geq 1$,

$$\partial_x \ \frac{U^{n+1}}{n+1} = \partial_t \ \frac{U^n}{n}.$$

Removing the intervening steps, we have

$$\partial_t^k \ U(x,t) = \partial_x^k \ \frac{(U(x,t))^{k+1}}{k+1}.$$

For me, this lies at the heart of the connection between Lagrange inversion and the IBHE for

$$\partial_t^k \ U = \partial_x^k \ \frac{U^{k+1}}{k+1}$$

implies (diverging from Buchstaber and Panov's presentation)

$$e^{\alpha \ \partial_t} \ U(x,t) = U(x,t+\alpha) = \ \sum_{k\geq 0} \ \alpha^k \frac{\partial_x^k}{k!} \ \frac{[U(x,t)]^{k+1}}{k+1}.$$

Spot checks: use

$$U(x,t) = \bar{U}_R(x,-t) = Cat(x,t) = -\frac{x-\frac{-1+\sqrt{1-4xt}}{2t}}{t}$$

$$= x^2 + 2tx^3 + 5t^2x^4 + 14t^3x^5 + \cdots \ ,$$

and expand both sides of the equation as series after action with $\partial_{\alpha=0}^p$.

From my post "The Lagrange Reversion Theorem and Inversion Formula Revisited"

$$v(x,y) = x + \sum_{k\geq 0} \frac{\partial_x^k}{k!} \; \frac{[x - v^{(-1)}(x,y)]^{k+1}}{k+1},$$

or

$$\frac{v(x,y)-x}{y} = \sum_{k\geq 0} \; y^k \; \frac{\partial_x^k}{k!} \; \frac{[\frac{x-v^{(-1)}(x,y)}{y}]^{k+1}}{k+1},$$

with the compositional inversion w.r.t. x, so, identifying $y = -t$ and $v(x,y) = M(x,t)$, this last identity becomes

$$\frac{M(x,-t)-x}{-t} = \sum_{k\geq 0} (-t)^k \quad \frac{\partial_x^k}{k!} \; \frac{[\frac{x-M^{(-1)}(x,-t)}{-t}]^{k+1}}{k+1},$$

which, from the formulas above in the discussion on the IBHE, is equivalent to

$$F(x) = U(x,0) = \sum_{k\geq 0} (-t)^k \; \frac{\partial_x^k}{k!} \; \frac{[U(x,-t)]^{k+1}}{k+1},$$

in agreement with

$$U(x,t+\alpha) \mid_{\alpha=-t} = \sum_{k\geq 0} \; \alpha^k \; \frac{\partial_x^k}{k!} \; \frac{[U(x,\alpha)]^{k+1}}{k+1} \; \mid_{\alpha=-t},$$

showing how Lagrange inversion is related to shifts in time of solutions to the IBHE.

Then, explicitly, for a general formal series for $F(x) = U(x,0)$,

$$(m_1x^2 + m_2x^3 + m_4x^5 + \cdots)$$

$$= \sum_{k\geq 0}(-t)^k \; \frac{\partial_x^k}{k!}[m_1x^2 + (2m_1^2t + m_2)x^3 + (5m_1^3t^2 + 5m_1m_2t + m_3)x^4$$

$$+(14m_1^4t^3 + 21m_1^2m_2t^2 + (6m_1m_3 + 3m_2^2)t + m_4)x^5 + \cdots]^{k+1}/(k+1).$$

To corroborate that U satisfying

$$e^{\alpha\,\partial_t} \, U(x,t) = U(x,t+\alpha) = \sum_{k\geq 0} \quad \alpha^k \frac{\partial_x^k}{k!} \; \frac{[U(x,t)]^{k+1}}{k+1},$$

solves the IBHE, act on the equation with $\partial_{\alpha=0}$, giving

$\partial_t\, U(x,t) = \partial_x\, \frac{[U(x,t)]^2}{2} = U(x,t)\partial_x\, U(x,t),$

showing the intimate relations among compositional inverse pairs, an avatar of the Lagrange inversion expansion formula, the inviscid Hopf-Burgers equation, and the refined Euler characteristic partition polynomials of the associahedra. However, the associahedra or not the only associated combinatorial constructs. To stress once more, the Lagrange expansion can be done with different series reps--power series, or o.g.f.s (A133437); Taylor series, or e.g.f.s (A134685); a log rep (/A133932); or a shifted reciprocal rep (A134264), all with different associated combinatorial constructs (see A145271 for a general prescription). Except for the shifted reciprocal rep, these different reps morph into each other with simple scalings of the indeterminates, just as an o.g.f. into its associated e.g.f.

Spot check for $m_1 = 1$ and $m_n = 0$ otherwise:

The last equation reduces to

$x^2 = \sum_{k\geq 0}(-t)^k\, \frac{\partial_x^k}{k!}\, \frac{[x^2+2tx^3+5t^2x^4+14t^3x^5+\cdots]^{k+1}}{k+1} \;= \sum_{k\geq 0}(-t)^k\, \frac{\partial_x^k}{k!}\, \frac{[Cat(x,t)]^{k+1}}{(k+1)},$

from the o.g.f. of the Catalan sequence above.

Evaluating the RHS for $t = 0$, gives $x^2 = x^2$.

Taking the derivative w.r.t. t on both sides and evaluating at $t = 0$ leads to

$0 = -\partial_x\, \frac{[Cat(x,0)]^2}{2} + \partial_{t=0}\, Cat(x,t),$

which is valid since

$\partial_x\, \frac{[Cat(x,0)]^2}{2} = \partial_x\, \frac{[x^2]^2}{2} = 2x^3$

and

$\partial_{t=0}\, Cat(x,t) = 2x^3.$

Hand-check that both sides are nulled by $\partial^2_{t=0}$ as well.

Faces of the associahedra

We could just as well choose $v(x,t) = M(x,t)$ in the last manifestation of the Lagrange inversion, giving

$$\frac{M^{(-1)}(x,-t)-x}{-t} = \sum_{k\geq 0} (-t)^k \ \frac{\partial_x^k}{k!} \ \frac{[\frac{x-M(x,-t)}{-t}]^{k+1}}{k+1} = \sum_{k\geq 0} (-t)^k \ \frac{\partial_x^k}{k!} \ \frac{[-F(x)]^{k+1}}{k+1},$$

or

$$\frac{M^{(-1)}(x,-t)-x}{t} = \sum_{k\geq 0} \ t^k \ \frac{\partial_x^k}{k!} \ \frac{[F(x)]^{k+1}}{k+1},$$

equivalent to

$$m_1x^2 + (2m_1^2t + m_2)x^3 + (5m_1^3t^2 + 5m_1m_2t + m_3)x^4$$

$$+(14m_1^4t^3 + 21m_1^2m_2t^2 + (6m_1m_3 + 3m_2^2)t + m_4)x^5 + \cdots$$

$$= \sum_{k\geq 0} \ t^k \ \frac{\partial_x^k}{k!} \ \frac{[m_1x^2+m_2x^3+\cdots]^{k+1}}{k+1},$$

implying, for $k \geq 1$,

$$\partial_{t=0}^k \ [x + m_1x^2 + (2m_1^2t + m_2)x^3 + (5m_1^3t^2 + 5m_1m_2t + m_3)x^4 + \cdots]$$

$$= \partial_x^k \ \frac{[m_1x^2+m_2x^3+\cdots]^{k+1}}{k+1}.$$

The monomial t^n flags the partitions corresponding to faces of the associahedra; e.g., the 2-D associahedra is composed of 5 zero-Dim vertices, 5 one-Dim edges, and 1 two-Dim pentagon and the associated partition polynomial is $5m_1^3t^2 + 5m_1m_2t + m_3$, so the partial derivatives w.r.t. t evaluated a $t = 0$ are selecting the faces of the same codimension (i.e., dimension of associahedron minus n, for $n = 1$, these are the facets) of each associahedron, multiplying them by the same factor, and nulling out the other faces. Then the LHS of the equation with all the moments set to unity is proportional to an o.g.f. enumerating the same dimensional faces over all the associahedra. For example, with $k = 1$, then

$$\partial_x \frac{(\frac{x^2}{1-x})^2}{2} = 2x^2 + 5x^3 + 9x^4 + \cdots$$

generates the number of facets of each associahedra given in the second column of the number triangle A033282, third column of A086810, and the second diagonal of A126216. The sequence is A000096.

The compositional inverse of

$$M(x,t) = x - tF(x) = x - t\frac{x^2}{1-x}$$

is

$$M^{(-1)}(x,t) = \frac{1+x-\sqrt{(1-x)^2-4xt}}{2(1+t)}$$

$$= x + tx^2 + (2t^2+t)x^3 + (5t^3+5t^2+t)x^4 + (14t^4+21t^3+9t^2+t)x^5$$

$$+(42t^5+84t^4+56t^3+14t^2+t)x^6 + \cdots,$$

a generator for the augmented number triangle A086810, a mirror image of A033282. The set of notes "Generators, Inversion, and Matrix, Binomial, and Integral Transforms" sketches the relations among number triangles derived from compositional inverse pairs of this type, reverts of the triangles, and certain types of binomial transforms.

With $k=2$,

$\partial_x^2 \frac{(\frac{x^2}{1-x})^3}{3!} = 5x^3 + 21x^4 + 56x^3 + \cdots$, generating the third column of A033282, fourth column of A086810, and the third diagonal of A126216. The sequence is A033275.

In general, $\partial_x^{n-1} \frac{(\frac{x^2}{1-x})^n}{n!}$

generates the n-th column of A033282 and the n-th diagonal of A126216.

Powers of the inverse and derivation by the indeterminates

From earlier arguments,

$$\partial_{m_n} \frac{(M^{(-1)}(x))^{k(n+1)}}{k} = -\partial_x \frac{(M^{(-1)}(x))^{(k+1)(n+1)}}{k+1},$$

or

$$\partial_{m_n} \frac{H^k}{k} = -\partial_x \frac{H^{(k+1)}}{k+1},$$

with $H(x) = (M^{(-1)}(x))^{n+1}$, so, repeating the maneuvers done above on U with U replaced by H and t by $-m_n$,

$$\frac{\partial^k_{m_n}}{k!}\ H = (-1)^k\ \frac{\partial^k_x}{k!}\ \frac{H^{k+1}}{k+1}$$

and

$$e^{\alpha\ \partial_{m_n}}\ H(x; m_1, m_2.., m_n, ..) = H(x; m_1, m_2.., m_n + \alpha, ..)$$

$$= \sum_{k\geq 0}(-\alpha)^k \frac{\partial^k_x}{k!} \frac{H(x; m_1, .., m_n, ..)^{k+1}}{k+1},$$

or

$$e^{\alpha\ \partial_{m_n}}\ (M^{(-1)}(x; m_1, .., m_n, ..))^{n+1} = (M^{(-1)}(x; m_1, .., m_n + \alpha, ..))^{n+1}$$

$$= \sum_{k\geq 0}\ (-\alpha)^k \frac{\partial^k_x}{k!}\ \frac{(M^{(-1)}(x; m_1, .., m_n, ..))^{(n+1)(k+1)}}{k+1}.$$

Action on both sides with $\partial_{\alpha=0}$ gives

$$\partial_{m_n}\ (M^{(-1)}(x; m_1, .., m_n, ..))^{n+1} =\ -\partial_x \frac{(M^{(-1)}(x; m_1, .., m_n, ..))^{2(n+1)}}{2},$$

and expanding out (suppressing other indeterminates) gives

$$(n+1)\ M^{(-1)}(x; m_n))^n\ \partial_{m_n}\ M^{(-1)}(x; m_n)$$

$$= -(n+1)(M^{(-1)}(x; m_n))^{2n+1}\partial_x\ M^{(-1)}(x; m_n)$$

reconstructing our fundamental diff id

$$\partial_{m_n}\ M^{(-1)}(x; m_n) = -(M^{(-1)}(x; m_n))^{n+1}\ \partial_x\ (M^{(-1)}(x; m_n))$$

$= -\partial_x \frac{(M^{(-1)}(x;m_n))^{n+2}}{n+2}.$

In particular, setting $\alpha = -m_n$,

$$(M^{(-1)}(x;m_1,..,m_n=0,..))^{n+1} = \sum_{k\geq 0} m_n^k \frac{\partial_x^k}{k!} \frac{(M^{(-1)}(x;m_1,..,m_n,..))^{(n+1)(k+1)}}{k+1}.$$

Returning to

$$e^{\alpha\,\partial_{m_n}} H(x;m_1,m_2..,m_n,..) = H(x;m_1,m_2..,m_n+\alpha,..)$$

$$= \sum_{k\geq 0} (-\alpha)^k \frac{\partial_x^k}{k!} \frac{H(x;m_1,..,m_n,..)^{k+1}}{k+1}$$

and acting on it with $\partial_{\alpha=0}$ gives

$$\partial_{m_n} H(x;m_n) = -\partial_x \frac{H(x;m_n)^2}{2} = -H(x;m_n)\partial_x H(x;m_n),$$

an IBHE with m_n a proxy for t. We've already seen that if H is a solution of the IBHE then it must satisfy the diff ids used to construct the associated exponential equation, so this is self-consistent.

I've shown in my pdf on the IBHE how to construct the general solution $U(x,t)$ for

$U(x,t=0) = F(x) = \sum_{k>1} b_k x^k$, and

$$H(x,\bar{t};m_1,m_2,..,m_n=t=0,..) = (M^{(-1)}(x,\bar{t}))^{n+1} \,|_{m_n=t=0}$$

for $n \geq 1$ as well as for $n=0$ with $m_0 = \bar{t} = t = 0$ are such a series, and this implies the validity of the diff ids proposed at the start of this set of notes;

$$\partial_{m_1} C(x) = x\partial_x M^{(-1)}(x),$$

$$\partial_{m_1} \frac{C(x)}{x} = \partial_{m_1} \frac{1}{M^{(-1)}(x)} = \partial_x M^{(-1)}(x) = \partial_x \frac{x}{C(x)},$$

$$\partial_{m_1} M^{(-1)}(x) = -\partial_x \frac{(M^{(-1)}(x))^3}{3},$$

and, more generally,

$$\partial_{m_n}\ M^{(-1)}(x) = -\partial_x\ \frac{(M^{(-1)}(x))^{n+2}}{n+2}.$$

This validates the results of the Borel-Laplace transform approach.

In a forthcoming set of notes, I'll characterize the relations between generalized Hermite diff ops of the sort $h(z) + g(z)\partial_z$ and conjugation, flow functions, and compositional inversion and hence the IBHE. This is related to the Virasoro-Heisenberg algebra, Schwarz-Kac diff ops, the KdV and KP evolution equations and certain quantum field models.

Appendix: Laurent series formulation

See "Three lectures on free probability" by Novak and LaCroix for a rationale for the Laurent series formulation--the desire to relate the free moments to a probability density function and hence to a characteristic function based on the Cauchy / Stieltjes transform.rather than the Fourier / Laplace transform based characteristic function for the classic moments--a free o.g.f. analog to the classic e.g.f. formulation. The Cauchy-Stieltjes transform approach has certain associations to Riemann mappings as well.

The Laurent series relations among the formal free cumulants and free moments is

$$LC(z) = \frac{C(z)}{z} = \frac{1}{O(z)} = \frac{1}{M^{(-1)}(z)} = \frac{1}{z} + c_1 + c_2 z + c_3 z^2 + \cdots$$

$$= \frac{1}{z} + m_1 + (m_2 - m_1^2)z + (m_3 - 3m_2 m_1 + 2m_1^3)z^2 + \cdots\ ,$$

with the formal compositional inverse

$$LC^{(-1)}(z) = LM(z) = M(\tfrac{1}{z}) = O^{(-1)}(\tfrac{1}{z}) = \frac{1}{z} + \frac{m_1}{z^2} + \frac{m_2}{z^3} + \frac{m_3}{z^4} + \cdots$$

$$= \frac{1}{z} + \frac{c_1}{z^2} + \frac{c_2 + c_1^2}{z^3} + \frac{c_3 + 3\ c_2 c_1 + c_1^3}{z^4} + \cdots\ .$$

Because the free cumulant partition polynomials are an Appell sequence in c_1,

$$\partial_{c_1} LC^{(-1)}(z) = \partial_{c_1}\ (\frac{1}{z} + \frac{c_1}{z^2} + \frac{c_2 + c_1^2}{z^3} + \frac{c_3 + 3\ c_2 c_1 + c_1^3}{z^4} + \cdots)$$

$= \frac{1}{z^2} + \frac{2c_1}{z^3} + \frac{3(c_2+c_1^2)}{z^4} + \frac{4(c_3+3\ c_2c_1+c_1^3)}{z^5} + \cdots$

$= -\partial_z\ LC^{(-1)}(z).$

In addition, from the diff ids proved above,

$\partial_{m_1} LC(z) = -(LC(z))^2 \partial_{m_1} M^{(-1)}(z)$

$= \partial_{m_1}\ (\frac{1}{z} + m_1 + (m_2 - m_1^2)z + (m_3 - 3m_2m_1 + 2m_1^3)z^2 + \cdots)$

$= 1 + 2(-m_1)z + 3(2m_1 - m_2)z^2 + \cdots = \partial_z M^{(-1)}(z),$

giving, self-consistently,

$\partial_{m_1}\ M^{(-1)}(z) = -\frac{1}{(LC(z))^2}\ \partial_z M^{(-1)}(z)$
$= -(M^{(-1)}(z))^2\ \partial_z\ M^{(-1)}(z) = -\partial_z \frac{(M^{(-1)}(z))^3}{3}.$

In a forthcoming set of notes, I'll prove the inverse relation between the two Laurent series via a flow function.

Related Stuff:

Abd-el-Malek and El-Mansib, "Group theoretic methods applied to Burgers' equation", for some history on the IBHE

Adler, "Set partitions and integrable hierarchies"

Aswin, Awasthi, Bonkile, Lakshmi, and Mukundan, "A systematic literature review of Burgers' equation with recent advances", for some history on the IBHE

Dykema, Nica, and Voiculescu, "Free Random Variables", on the IBHE

Dubrovin and Yang, "Remarks on intersection numbers and integrable hierarchies. I. Quasi-triviality", see the Riemann hierarchy / diff ids of eqn. 2.3 on p. 4.

Kenyon and Okounkov, “Limit shapes and the complex Burgers equation”

Matytsin, "On the Large N Limit of the Itzykson-Zuber Integral" has a beautiful presentation of the complex inviscid Burgers equation, which separates into two real equations--the transport and the IBHE.

Menon, “The complex Burgers equation, the HCIZ integral and the Calogero-Moser system”

Olver, course notes “Chapter 22. Nonlinear Partial Differential Equations” and his book “Introduction to Nonlinear Partial Differential Equations”: a conservation law is an equation of the form $\partial_x T + \partial_x X = 0,$ and T and X are the conserved density and associated flux, respectively.

Voiculescu, “Addition of Certain Non-commuting Random Variables”, for IBHE