A Taste of Moonshine in Free Moments:

Cumulants and moments of free probability, noncrossing partitions, and inversions of Laurent series

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This set of notes collates the differing notation and identities found in earlier notes of mine and papers by other researchers related to a core set of relationships among formal power series, associated Laurent series, and their multiplicative and compositional inverses. Concrete applications of these identities are given--one of the most interesting, for the q-expansion of the Klein j-invariant, associated with monster moonshine.

(Since this is sort of a Rosetta stone for translating among different presentations of related constructs, there will be much redundancy in the expressions. See the appendix on the notation for designating the compositional inverse and the reciprocal, i.e., the multiplicative inverse.)

Consider the power series, or ordinary generating function (o.g.f.),

$$\omega(z) = z + \alpha_1 \; z^2 + \alpha_2 \; z^3 + \cdots$$

its shifted reciprocal

$$\overline{\omega}(z) = \frac{z}{\omega(z)} = \frac{1}{1 \, + \, \alpha_1 \;\, z \, + \, \alpha_2 \;\, z^2 \, + \, \cdots} = 1 + \beta_1 \; z + \beta_2 \; z^2 + \cdots \; ,$$

and the reciprocal of its derivative

$$g(z) = \frac{1}{D_z \omega(z)} = \frac{1}{D_z \frac{z}{\overline{\omega}(z)}} = \frac{1}{D_z \frac{z}{1 \, + \, \beta_1 \;\, z \, + \, \beta_2 \;\, z^2 \, + \cdots}} .$$

in terms of the coefficients β_n.

Then

$$\exp[z\ g(u)D_u]\ u = \exp[z\ \tfrac{d}{d\omega(u)}]\ \omega^{(-1)}(\omega(u)) = \omega^{(-1)}(\omega(u)+z),$$

so

$$\exp[z\ g(u)D_u]\ u\ |_{u=0} = \omega^{(-1)}(z) = z + \gamma_1\ z^2 + \gamma_2\ z^3 + \cdots\ ,$$

and, the formal Taylor series coefficients of the compositional inverse are generated by

$$D_z^n\ \omega^{(-1)}(z)\ |_{u=0} = [g(u)D_u]^n\ u\ |_{u=0}$$

whereas the coefficients of the formal power series of the inverse are given by

$$\gamma_n = \tfrac{D_z^n}{n!}\ \omega^{(-1)}(z)\ |_{u=0} = \tfrac{1}{n!}\ [g(u)D_u]^n\ u\ |_{u=0}.$$

With different reps for $g(z)$, we can generate diverse sets of partition polynomials--each set with its own set of combinatorial interpretations--for the coefficients of the inverse. One set of Lagrange inversion partition polynomials and links (in the Cf. section of the OEIS entry) to several other sets are provided in OEIS A145271.

The inversion partition polynomials in terms of the indeterminates $\beta_n = h_n$ with $h_0 = 1$ are illustrated in A134264 and are intimately associated with noncrossing partitions, parking functions, Dyck lattice paths, and various other combinatorial constructs. They are called the Voiculescu polynomials by free probabilists. The iterated Lie generator above provides a method of quickly generating the inversion partition polynomials to low orders via a symbolic math app with the result

$$\gamma_1 = \beta_1 = Prt_1(1, \beta_1),$$

$$\gamma_2 = \beta_2 + \beta_1^2 = Prt_2(1, \beta_1, \beta_2),$$

$$\gamma_3 = \beta_3 + 3\ b_2\beta_1 + \beta_1^3 = Prt_3(1, \beta_1, \beta_2, \beta_3),$$

and

$$\gamma_4 = \beta_4 + 2\ \beta_2^2 + 4\ \beta_3\ \beta_1 + 6\ \beta_2\ \beta_1^2 + \beta_1^4 = Prt_4(1, \beta_1, \beta_2, \beta_3, \beta_4).$$

This is an Appell Sheffer polynomial sequence in the indeterminate β_1 treated as a continuous variable, so

$$\frac{d}{d\beta_1}\ Prt_n(1,\beta_1,...,\beta_n) = n\ Prt_{n-1}(1,\beta_1,...,\beta_{n-1})$$

provides a quick sanity check of the polynomial sequence.

Reprising, the formal inverse about the origin of the formal power series

$$\omega(z) = z + \alpha_1\ z^2 + \alpha_2\ z^3 + \cdots$$

with the formal shifted reciprocal

$$\overline{\omega}(z) = \frac{z}{\omega(z)} = 1 + \beta_1\ z + \beta_2\ z^2 + \cdots$$

is

$$\omega^{(-1)}(z) = (\frac{z}{\overline{\omega}(z)})^{(-1)} = z + \beta_1\ z^2 + (\beta_2 + \beta_1^2)\ z^3 + (\beta_3 + 3\ b_2\beta_1 + \beta_1^3)\ z^4$$

$$+ (\beta_4 + 2\ \beta_2^2 + 4\ \beta_3\ \beta_1 + 6\ \beta_2\ \beta_1^2 + \beta_1^4)\ z^5 + \cdots .$$

I'll refer to this as ***the noncrossing partition inversion algorithm (NCPIA)***.

Substitutions giving simple identities among the inverses and reciprocals such as

$$\bar{\omega}(\omega^{(-1)}(z)) = \frac{z}{\omega(z)}\ |_{z\to\omega^{(-1)}(z)} = \frac{\omega^{(-1)}(z)}{z}$$

or, conversely,

$$\frac{\omega^{(-1)}(z)}{z}\ |_{z\to\omega(z)} = \frac{z}{\omega(z)} = \bar{\omega}(z)$$

are used in morphed forms throughout discussions of free probability, as we'll see below. I'll refer to these as ***the inverse reciprocal substitution identities (IRSI)***.

Another useful, but tricky, identity is

$$(\frac{1}{K(\frac{1}{z})})^{(-1)} = \frac{1}{K^{(-1)}(\frac{1}{z})}$$

for

$\frac{1}{K(\frac{1}{z})} \mid_{z \to \frac{1}{K^{(-1)}(\frac{1}{z})}} = \frac{1}{K(K^{(-1)}(\frac{1}{z}))} = z$;

i.e., one has to be careful in pairing up mutually inverse bijective functions over the domains of z of interest. I'll refer to this as ***the inverse reciprocal identity (IRI)***. This is illustrated in an example below. Note, as stressed in the Appendix, that inversion and substitution do not commute in general, so it's important to distinguish between, e.g., $(f(\frac{1}{x}))^{(-1)} = (h(x))^{(-1)} = h^{(-1)}(x)$, with $h(x) = f(\frac{1}{x})$, and $f^{(-1)}(\frac{1}{x}) = (f(u))^{(-1)} \mid_{u=\frac{1}{x}}$.

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Now define the Laurent series

$$f(z) = z + \beta_1 + \frac{\beta_2}{z} + \frac{\beta_3}{z^2} + \cdots = z\,\overline{\omega}(\tfrac{1}{z}).$$

Then

$$\overline{\omega}(z) = zf(\tfrac{1}{z}) = 1 + \beta_1 z + \beta_2 z^2 + \cdots$$

and the NCPIA and RII imply

$$\omega^{(-1)}(z) = (\tfrac{z}{\overline{\omega}(z)})^{(-1)} = (\tfrac{1}{f(\frac{1}{z})})^{(-1)} = \tfrac{1}{f^{(-1)}(\frac{1}{z})}.$$

Define

$$\tilde{f}(z) = \omega^{(-1)}(z) = z + \beta_1\, z^2 + (\beta_2 + \beta_1^2)\, z^3 + (\beta_3 + 3\, b_2\beta_1 + \beta_1^3)\, z^4 + \cdots.$$

Then

$$\tilde{f}(z) = (\tfrac{z}{\overline{\omega}(z)})^{(-1)} = (\tfrac{1}{f(\frac{1}{z})})^{(-1)} = \tfrac{1}{f^{(-1)}(\frac{1}{z})},$$

which I'll refer to as ***the hybrid inverse reciprocal identity (HIRI)*** since it combines the NCPIA and IRI,

and

$$f(\tfrac{1}{z}) \mid_{z \to [\frac{1}{f^{(-1)}(z)} = \tilde{f}(\frac{1}{z})]} = f(f^{(-1)(z)}) = z,$$

so

$$(f(\tfrac{1}{z}))^{(-1)} = \tilde{f}(\tfrac{1}{z}).$$

Consequently, the Laurent series

$$L(z) = f(\tfrac{1}{z}) = \tfrac{1}{z} + \beta_1 + \beta_2\, z + \beta_3\, z^2 + \cdots$$

is inverse to the Laurent series

$$L^{(-1)}(z) = \tilde{f}(\tfrac{1}{z}) = \tfrac{1}{z} + \tfrac{\beta_1}{z^2} + \tfrac{\beta_2+\beta_1^2}{z^3} + \tfrac{\beta_3+3\, b_2\beta_1+\beta_1^3}{z^4} + \cdots$$

$$= \tfrac{1}{z} + \tfrac{Prt_1(1,h_1)}{z^2} + \tfrac{Prt_2(1,h_1,h_2)}{z^3} + \cdots .$$

I'll refer to this as ***the Laurent series inversion identity (LSII)***.

The identity is also presented in He and Jejjala (see refs just below). $L(z)$ has the form of the celebrated q-expansion of Klein's j-invariant, which plays a central role in the formalism of monstrous moonshine and modular functions, as discussed in detail by H & J, hence, the allusion in the title.

This concludes a review of core identities linking inverses and reciprocals found in the presentations listed in the next section.

Below I'll re-state and/or re-prove some of the identities above in the differing notation of the presentations by various authors and illustrate their application.The earlier notes to which I'll mainly refer below by author are

1) "Appell polynomials, cumulants, noncrossing partitions, Dyck lattice paths, and inversion" by myself, posted at my WordPress math blog Shadows of Simplicity
2) "Multiplicative functions on the non-crossing partitions and free convolution" by Roland Speicher
3) "Appell polynomials and their relatives" by Michael Anshelevich
4) "Enumerative geometry, tau-functions and Heisenberg–Virasoro algebra" by Alexander Alexandrov
5) "Modular Matrix Models" by Yang-Hui He and Vishnu Jejjala (H & J)

Speicher, on p. 616, defines

$$A(z) = 1 + \sum_{n \geq 1} \ a_n \ z^n$$

and

$$B(z) = 1 + \sum_{n \geq 1} \ b_n \ z^n,$$

and asserts from relations of the coefficients to noncrossing partitions that

$$A(zB(z)) = B(z)$$

and

$$B(\tfrac{z}{A(z)}) = A(z),$$

so

$$zA(zB(z)) = zB(z).$$

These identities are easily related to morphs of the obvious IRSI in the initial section above after re-stating them in terms of inverses of power series, and the relation to the NCPIA follows.

Let $W(z) = zB(z)$, then $A(zB(z)) = B(z)$ implies

$$zA(W(z)) = W(z)$$

and

$$W^{(-1)}(z)A(z) = z,$$

or

$$A(z) = \tfrac{z}{W^{(-1)}(z)}.$$

Consequently, $B(\tfrac{z}{A(z)}) = A(z)$ implies

$B(W^{(-1)}(z)) = \frac{z}{W^{(-1)}(z)},$

so

$B(z) = \frac{W(z)}{z}.$

Then $W(z) = zB(z) = z + \sum_{n\geq 1}\ b_n\ z^{n+1},$

$A(z) = \frac{z}{W^{(-1)}(z)}$,

and

$B(z) = \frac{W(z)}{z}$

imply

$A(zB(z)) = A(W(z)) = \frac{z}{W^{(-1)}(z)}\ |_{z\to W(z)} = \frac{W(z)}{z} = B(z)$

and

$B(\frac{z}{A(z)}) = B(W^{(-1)}(z)) = \frac{W(z)}{z}\ |_{z\to W^{(-1)}(z)} = \frac{z}{W^{(-1)}(z)} = A(z).$

These last two identities have the IRSI embedded in them.

Beginning directly wIth

$W(z) = zB(z) = z + \sum_{n\geq 1}\ b_n\ z^{n+1}$

and

$A(z) = \frac{z}{W^{(-1)}(z)} = 1 + \sum_{n\geq 1}\ a_n\ z^n$,

the NCPIA directly conjures up the Voiculescu polynomials, such as

$b_1 = a_1,$

$b_2 = a_2 + a_1^2,$

$b_3 = a_3 + 3\ a_2 a_1 + a_1^3,$

and

$b_4 = a_4 + 2\ a_2^2 + 4 a_3\ a_1 + 6\ a_2 a_1^2 + a_1^4.$

The other identities in Speicher noted above follow directly from the identities in the earlier sections

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To relate the above to the presentation in my blog post, identify

$O_m(z) = zB(z) = W(z) = z + b_1 z^2 + b_2 z^3 + \cdots\ ,$

$\frac{O_r(z)}{z} = A(z) = \frac{z}{W^{(-1)}(z)} = \frac{z}{O_m^{(-1)}(z)}$

$= 1 + RT(z) = 1 + a_1 z + a_2 z^2 + \cdots\ ,$

where $RT(z)$ is the series for the R transform developed by Voiculescu, and

$G(z) = O_m(\frac{1}{z}) = \frac{B(\frac{1}{z})}{z} = W(\frac{1}{z})$

$= \frac{1}{z} + b_1 \frac{1}{z^2} + b_2 \frac{1}{z^3} + \cdots\ ,$

a Laurent series of the moments of a pdf, or formal moments, introduced by Speicher on p. 226 and by Anshelevich on p. 17.

Then

$G(\frac{1+RT(z)}{z}) = O_m(\frac{z}{1+RT(z)}) = O_m(O_m^{(-1)}(z)) = z,$

implying

$G^{(-1)}(z) = \frac{1+RT(z)}{z} = \frac{1}{O_m^{(-1)}(z)}$

$$= \frac{1}{W^{(-1)}(z)} = \frac{O_r(z)}{z^2}$$

and together with

$$G^{(-1)}(z) = (O_m(\tfrac{1}{z}))^{(-1)} = (W(\tfrac{1}{z}))^{(-1)}$$

that

$$\frac{1}{O_m^{(-1)}(z)} = (O_m(\tfrac{1}{z}))^{(-1)},$$

or

$$O_m^{(-1)}(z) = \frac{1}{(O_m(\frac{1}{z}))^{(-1)}},$$

or

$$(O_m^{(-1)}(z))^{-1} = (O_m(z^{-1}))^{(-1)}.$$

Alternatively, the compositional inverse of both sides gives

$$(\frac{1}{O_m^{(-1)}(z)})^{(-1)} = O_m(\tfrac{1}{z}),$$

an identity that holds for any function analytic at the origin with a power series expansion of the form $O_m(z) = z + \sum_{n \geq 1} \; c_n \; z^n$ about the origin $z = 0$. This can be extended to a formal power series and its formal inverse.

The series of identities are obviously valid when couched as the equivalent statement

$$\frac{1}{O_m^{(-1)}(z)} \;|_{z \to O_m(\frac{1}{z})} = \frac{1}{O_m^{(-1)}(O_m(\frac{1}{z}))} = \frac{1}{\frac{1}{z}} = z$$

with the caveat that $|z|$ is large enough that $1/|z|$ lies within the radius of convergence of the power series for $O_m(z)$ about the origin. For z real, $O_m(z)$ is strictly increasing in a sufficiently small neighborhood about the origin, and a unique inverse exists in this neighborhood..

Checks, illustrations, and caveats:

Let

$O_m(z) = \frac{z}{1-z}$.

Then

$O_m^{(-1)}(z) = \frac{z}{1+z}$

and

$$\frac{1}{(O_m(\frac{1}{z}))^{(-1)}} = \frac{1}{(\frac{1}{z-1})^{(-1)}} = \frac{1}{\frac{1}{z}+1} = \frac{z}{1+z} = O_m^{(-1)}(z)$$

$$\neq \frac{1}{O_m^{(-1)}(\frac{1}{z})} = \frac{1}{O_m^{(-1)}(z)\mid_{z\to\frac{1}{z}}} = \frac{1}{\frac{z}{1+z}\mid_{z\to\frac{1}{z}}} = \frac{1}{\frac{1}{1+z}} = 1+z = \frac{1}{O_m^{(-1)}(z)}\mid_{z\to\frac{1}{z}}.$$

Substituting $1/z$ for z throughout the whole equation is not valid either since

$$\frac{1}{(O_m(z))^{(-1)}} = \frac{1}{(\frac{z}{1-z})^{(-1)}} = \frac{1}{\frac{z}{1-z}} = \frac{1-z}{z}$$

$$\neq \frac{1}{(O_m(\frac{1}{z}))^{(-1)}}\mid_{z\to\frac{1}{z}} = \frac{1}{\frac{1}{z}+1}\mid_{z\to\frac{1}{z}} = \frac{1}{z+1} = \frac{z}{1+z}\mid_{z\to\frac{1}{z}} = O_m^{(-1)}(\frac{1}{z}).$$

See the Appendix for more on caveats.

Returning to the main stream of associations, note then also

$$G^{(-1)}(z)|_{z\to\frac{1}{z}} = z(1+RT(\frac{1}{z}))$$

$$= \frac{1}{O_m^{(-1)}(z)}\mid_{z\to\frac{1}{z}} = zA(\frac{1}{z}) = z + a_1 + a_2 z^{-1} + a_3 z^{-2} + \cdots$$

$$= \frac{1}{W^{(-1)}(z)}|_{z\to\frac{1}{z}}.$$

Then identify in Alexandrov (p. 22) the Laurent series

$$f(z) = z + b_{-1} + b_{-2}z^{-1} + \cdots = z + a_1 + a_2 z^{-1} + a_3 z^{-2} + \cdots$$

$$= zA(\tfrac{1}{z}) = G^{(-1)}(z)|_{z\to\frac{1}{z}} = \tfrac{1}{W^{(-1)}(z)}|_{z\to\frac{1}{z}},$$

or

$$zf(\tfrac{1}{z}) = 1 + a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$

$$= A(z) = zG^{(-1)}(z) = \tfrac{z}{W^{(-1)}(z)} = \tfrac{O_r(z)}{z} = \tfrac{z}{O_m^{(-1)}(z)},$$

and the power series

$$\tilde{f}(z) = z + b_{-1}\ z^2 + (b_{-2} + b_{-1}^2)\ z^3 + (b_{-3} + 3\ b_{-1}b_{-2} + b_{-1}^3)\ z^4 + \cdots$$

$$= z + a_1\ z^2 + (a_2 + a_1^2)\ z^3 + (a_3 + 3\ a_1 a_2 + a_1^3)\ z^4$$

$$+(a_4 + 2\ a_2^2 + 4\ a_1 a_3 + 6\ a_1^2 a_2 + a_1^4)\ z^5 + \cdots$$

$$= z + b_1 z^2 + b_2 z^3 + b_3 z^4 + b_4 z^5 + \cdots$$

$$= zB(z) = W(z) = O_m(z).$$

Now let's get down to the nuts and bolts of a particular example of the application of the HIRI

$$\tilde{f}(z) = (\tfrac{z}{\overline{\omega}(z)})^{(-1)} = (\tfrac{1}{f(\frac{1}{z})})^{(-1)} = \tfrac{1}{f^{(-1)}(\frac{1}{z})},$$

a version of which is in Alexandrov on p. 22. There is some ambiguity in his notation for me, but I'll present more deeply the relationships among his diff ops, the compositional inverse, and his series in a forthcoming set of notes.

Consider a simple instance of a pair of functions analytic and mutually inverse about the origin

$$\tilde{f}_0(z) = \tfrac{1 - z - \sqrt{(1-z)^2 - 4z^2}}{2z} = W_0(z)$$

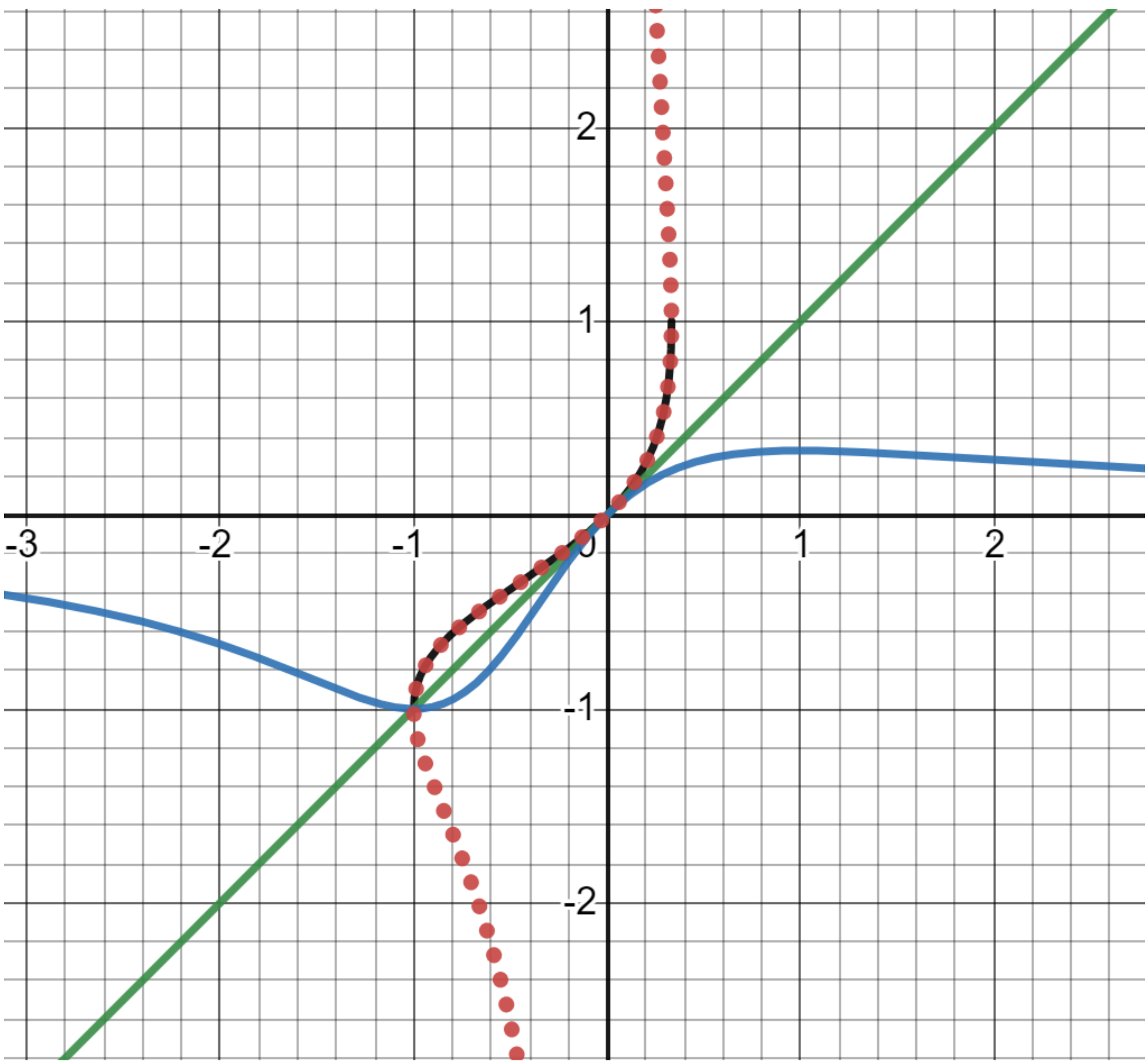
$$= z + z^2 + 2z^3 + 4z^4 + 9z^5 + 21z^6 + 51z^7 + \cdots ,$$

an o.g.f. for the Motzkin numbers A001006, discussed more generally in a OEIS A097610 ,

and

$$W_0^{(-1)}(z) = \frac{z}{1+z+z^2} = \frac{1}{z+1+\frac{1}{z}}.$$

In the following plot,



the curve for the global non-bijective function

$$y(x) = W^{(ref)}(x) = \frac{x}{1+x+x^2} = \frac{1}{x+1+\frac{1}{x}} = W^{(ref)}(\frac{1}{x})$$ is ***the blue curve***

while the curve for the mapping

$$x(y) = W^{ref}(y) = \frac{y}{1+y+y^2} = \frac{1}{y+1+\frac{1}{y}}$$ is ***the dotted red curve***,

which is not even a function and has a closed, connected finite domain. The two curves--the red $y = W(x)$ and the blue $y(x) = W^{(ref)}(x)$--are reflections of each other through the green bisector and are composed of local inverses for reflected strictly monotonically increasing or decreasing sections of each curve.

The real function with limited domain $-1 \leq x \leq \frac{1}{3}$ and range $-1 \leq y \leq 1$

$y(x) = \tilde{f}_0(x) = W_0(x) = \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x}$ Is ***the black section of finite length***

of the red dotted curve and is the inverse to an appropriate section of $y = W^{(ref)}(x)$, i.e.,

$$y(x) = \tilde{f}_0^{(-1)}(x) = W_0^{(-1)}(x) = \frac{x}{1+x+x^2} = \frac{1}{\frac{1}{x}+1+x} = W^{(ref)}(x) = W^{(ref)}(\frac{1}{x}),$$

the section of the blue curve for $-1 \leq x \leq 1$ and $-1 \leq y \leq \frac{1}{3}$.

Note that it would be invalid to substitute $W_0^{(-1)}(\frac{1}{x})$ for $W^{(ref)}(\frac{1}{x})$ since $W_0^{(-1)}(x)$ is defined as a bijective inverse only for $-1 \leq x \leq 1$.

In other words, $\tilde{f}_0(x) = W_0(x)$ is an analytic bijective rep of the strictly monotonically increasing section about the origin of the red curve

$$x(y) = W^{ref}(y) = \frac{y}{1+y+y^2} = \frac{1}{y+1+\frac{1}{y}} = W^{ref}(1/y)$$

in the xy-plane that is inverse to the strictly monotonically increasing section about the origin of the blue curve

$$y(x) = W^{ref}(x) = \frac{x}{1+x+x^2} = \frac{1}{\frac{1}{x}+1+x} = W^{ref}(1/x).$$

More comprehensively, denote this red monotonic section about the origin as $\tilde{f}_0(x)$, the red monotonic section above it as $\tilde{f}_+(x)$, and below it as $\tilde{f}_-(x)$. Then a complete rep of the inverse curve, call it $\tilde{f}(x)$ now, is the patchwork of analytic bijective functions

A) $y = \tilde{f}_+(x) = W_+(x) = \frac{1-x+\sqrt{(1-x)^2-4x^2}}{2x} = \frac{2x}{1-x-\sqrt{(1-x)^2-4x^2}} = \frac{1}{H(x))}$

for $0 < x \leq \frac{1}{3}$ and $y \geq 1$,

inverse to

$$y = W_+^{(-1)}(x) = \frac{x}{1+x+x^2} = \frac{1}{x+1+\frac{1}{x}} = W^{ref}(x)$$ for $x \geq 1$ and $0 \leq y \leq \frac{1}{3}$

B) $$y = \tilde{f}_0(x) = W_0(x) = \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x} = \frac{2x}{1-x+\sqrt{(1-x)^2-4x^2}} = H(x)$$

for $-1 \leq x \leq \frac{1}{3}$ and $-1 \leq y \leq 1$,

inverse to

$$y = W_0^{(-1)}(x) = \frac{x}{1+x+x^2} = W^{ref}(x)$$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq \frac{1}{3}$

C) $$y = \tilde{f}_-(x) = W_-(x) = \frac{1-x+\sqrt{(1-x)^2-4x^2}}{2x} = \frac{2x}{1-x-\sqrt{(1-x)^2-4x^2}} = \frac{1}{H(x)}$$

for $-1 \leq x < 0$ and $y \leq -1$

inverse to

$$y = W_-^{(-1)}(x) = \frac{x}{1+x+x^2} = W^{ref}(x)$$ for $x \leq -1$ and $-1 \leq y < 0$.

Note the symmetries induced among the inverse sections by the global symmetry

$$W^{ref}(x) = \frac{1}{x+1+\frac{1}{x}} = W^{ref}(\tfrac{1}{x})$$;

namely, $\tilde{f}_+(x)$ and $\tilde{f}_-(x)$ have the same analytic form $1/H(x)$ whereas $\tilde{f}_0(x)$ is analytically the reciprocal of $\tilde{f}_+(x)$ and $\tilde{f}_-(x)$, i.e., $H(x)$. Furthermore, the change in the signs of the square roots reflect

$$\tilde{f}_+(x)\tilde{f}_0(x) = \tilde{f}_-(x)\tilde{f}_0(x) = \frac{1}{H(x)}H(x) = 1$$.

We have now the nonmonotonic function, the blue curve, with the global analytic rep

$$\tilde{f}^{ref}(x) = W^{ref}(x) = \frac{x}{1+x+x^2} = \left(\frac{1}{\frac{1}{x}+1+x}\right)^{-1} = W^{ref}(\tfrac{1}{x}) = \tilde{f}^{ref}(\tfrac{1}{x})$$

and have obtained its local inverses for its three monotonic sections over which bijections can individually be established, so let’s check the applicability of the HIRI while also relating it to Alexandrov’s notation and a key identity in his paper.

For our special case,

$$zf(\tfrac{1}{z}) = \frac{z}{W^{ref}(z)} = 1 + z + z^2,$$

so

$$f(\tfrac{1}{z}) = f(z) = \frac{1}{W^{ref}(z)} = \frac{1+z+z^2}{z} = z + 1 + \frac{1}{z} = \frac{1}{\tilde{f}^{ref}(z)} = (\tilde{f}^{ref}(z))^{-1},$$

or

$$\tilde{f}^{ref}(z) = \frac{z}{1+z+z^2} = \frac{1}{\frac{1}{z}+1+z} = \frac{1}{f(\frac{1}{z})} = \frac{1}{f(z)},$$

for any finite real z other than $z = 0$.

Re-casting this in terms of compositional inverses is a bit tricky, but as it stands it does suggest along with the HIRI that, if we are careful enough about linking domains and ranges of the mappings, we can find bijective sections (with each section connected and strictly monotonically increasing or decreasing but separate multiple sections allowed) of the curves $y = \tilde{f}^{ref}(z)$ and $y = f(\frac{1}{z}) = \frac{1}{z} + 1 + z$ such that

$$W_S(z) = \tilde{f}_S(z) = \left(\frac{1}{f_S(\frac{1}{z})}\right)^{(-1)} = \frac{1}{f_S^{(-1)}(\frac{1}{z})} = \frac{1}{f_S^{(-1)}(z)\mid_{z\to\frac{1}{z}}},$$

and Alexandrov (p. 22) essentially claims, if my interpretation of his results and notation are valid but with some ambiguity as applied here at least, that, in general,

$$\tilde{f}(z) = \frac{1}{f^{(-1)}(\frac{1}{z})},$$

which is essentially what I called the HIRI above.

Then consider the bijective section, for $-1 \leq x \leq \frac{1}{3}$ and $-1 \leq y \leq 1$,

$$y = \tilde{f}_0(x) = W_0(x) = \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x} = \frac{1}{f_S^{(-1)}(\frac{1}{x})},$$

which, in turn, suggests

$f_S^{(-1)}(\frac{1}{x}) = f_S^{(-1)}(x)\mid_{x\to\frac{1}{x}} = \frac{1}{W_0(x)} = \frac{2x}{1-x-\sqrt{(1-x)^2-4x^2}}$ and

$f_S^{(-1)}(u) = \frac{2}{u-1-\sqrt{(1-u)^2-4}}$ for $u = \frac{1}{x}$ with $-1 \le x \le \frac{1}{3}$ and $-1 \le \frac{1}{f(u)} \le 1$, but this isn't correct as spot checks for $-1 \le x < 0$ would reveal.

The actual answer for this section $\tilde{f}_0(x)$ of the curve $y = \tilde{f}(x) = W(x)$ is that there are two different functions representing two different strictly monotonic sections of the curve $x = f(y) = \frac{1}{y} + 1 + y$, i.e., the curve $y = f^{ref}(x)$, whose analytic expressions are mutual reciprocals and which are bijective over different domains, that satisfy the HIRI

$$y = \tilde{f}_0(x) = W_0(x) = \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x} = \frac{1}{f_S^{(-1)}(\frac{1}{x})}.$$

One is the right blue curve

$$f_I^{(-1)}(x) = \frac{2}{x-1-\sqrt{(1-x)^2-4}} = C_B(x)$$

bijective for $x \ge 3$ and $y \ge 1$,

and the other, the left red curve

$$f_{IV}^{(-1)}(x) = \frac{2}{x-1+\sqrt{(1-x)^2-4}} = 1/C_B(x)$$

bijective for $x \le -1$ and $y \le -1$, which are local inverses of $y = f(x) = \frac{1}{x} + 1 + x$, i.e., appropriate sections of $y = f^{ref}(x)$, a.k.a. the curve $x = f(y) = \frac{1}{y} + 1 + y$.

In a section below, I show that for the particular curve here $y = \tilde{f}(x) = \frac{x}{1+x+x^2}$, the HIRI is satisfied piecemeal for the whole curve when we match appropriate bijective sections of the curve $y = \tilde{f}(x)$ with bijective sections of the reflection of $y = f(x) = \frac{1}{x} + 1 + x$, i.e., $x = f(y) = \frac{1}{y} + 1 + y$, which I denote by $y = f^{ref}(x)$ above, a mapping which doesn't have the same analytic expression over the complete curve. (Contrast the inverse pair of real functions over the reals $y = f(x) = e^x - 1$ and $y = f^{ref}(x) = f^{(-1)}(x) = \ln(1+x)$, bijective over their domains of definition, with the curves above.)

Checks:

$\tilde{f}_0(-\frac{1}{2}) \simeq -.38196$ and the left red curve gives $\frac{1}{f_{IV}^{(-1)}(-2)} \simeq -.38196$.

$\tilde{f}_0(\frac{1}{5}) \simeq .2679$ and the right blue curve gives $\frac{1}{f_{I}^{(-1)}(5)} \simeq .2679$.

However, note the analytic bijective function for the domain of the right blue curve is

$$f_I^{(-1)}(x) = \frac{2}{x-1-\sqrt{(1-x)^2-4}} = C_B(x)$$

and for the left red curve,

$$f_{IV}^{(-1)}(x) = \frac{2}{x-1+\sqrt{(1-x)^2-4}} = 1/C_B(x)$$

while

$$\tilde{f}_0(x) = \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x} .$$

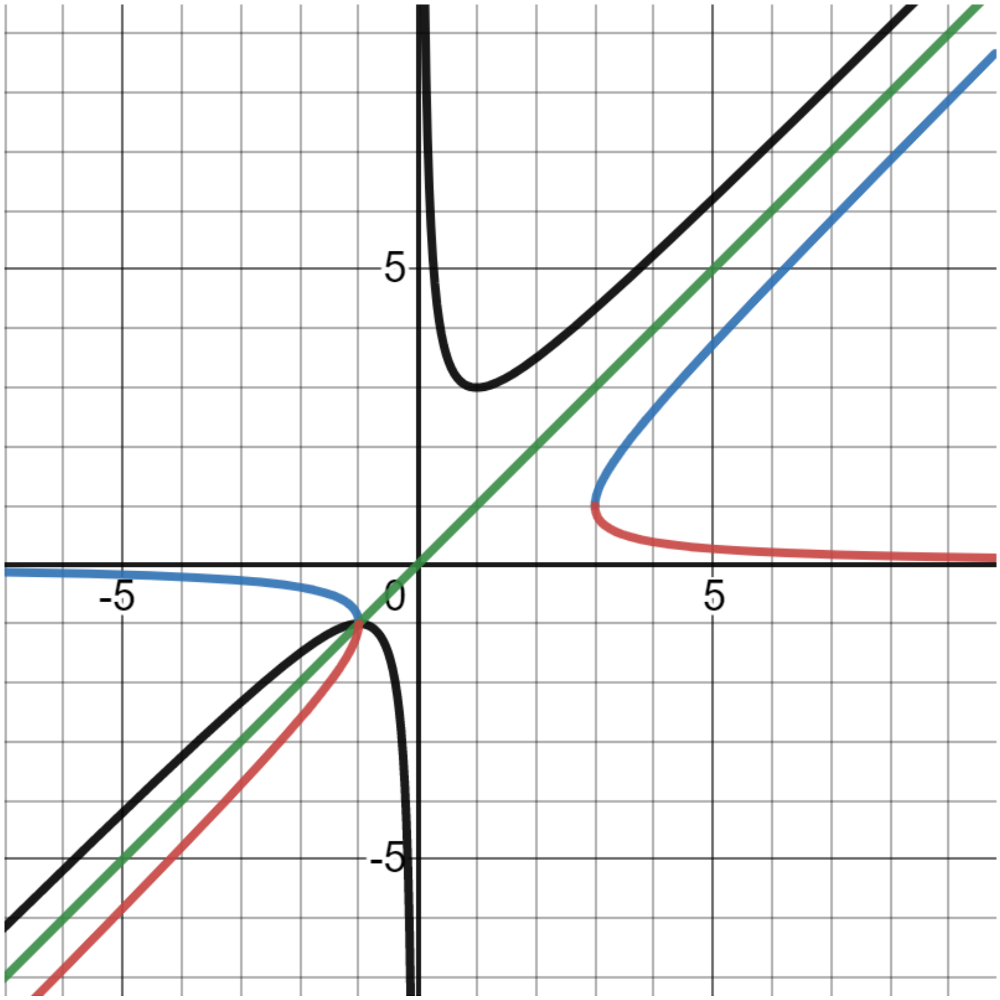
so, ignoring domains and ranges,

$$\frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x} = \frac{1}{C_B(\frac{1}{x})} \neq C_B(\frac{1}{x})$$

and it appears we have an apparent contradiction.This is an unjustified conclusion, rather $f_S^{(-1)}(\frac{1}{x})$ is a function discontinuous over both its domain and range, that cannot be represented fully by $C_B(1/x)$. The first hint of this is that the continuous connected domain $-1 \leq x \leq \frac{1}{3}$ of the bijective section $y = \tilde{f}_0(x)$ gets mapped to the two separate domains $x \leq -1$ and $x \geq 3$ under the transformation $x \to 1/x$. The reciprocal $1/f_S^{(-1)}(\frac{1}{x})$ must map into the continuous connected range $-1 \leq y \leq 1$ of $y = \tilde{f}_0(x)$ as well, so we can anticipate that the function $f_S^{(-1)}(\frac{1}{x})$ has two separate ranges $y \leq -1$ and $y \geq 1$ also.

The HIRI allows us to determine these curves from the local inverses of the curve $y = \tilde{f}(x) = W(x)$, and, conversely, the graph of the local inverses of $f(x)$ allows us to check the HIRI or determine the curve $y = \tilde{f}(x) = W(x)$, so now we will explore more fully the inverse pairs for the curve $y = f(x) = x + 1 + \frac{1}{x} = f(\frac{1}{x})$.

Choosing the positive value of the square root and, as above, using reflection through the bisector of the first and third quadrants of the xy-plane to define the local inverses of a curve, the reflected curve over the reals of $y = f(x) = x + 1 + \frac{1}{x}$ is characterized as a patchwork of four functions, each of which is either strictly monotonically increasing or decreasing. Accordingly, the following graph depicts four regions associated to the indices I, II, III, and IV for which bijective associations of sections of $y = f(x) = x + 1 + \frac{1}{x}$ with analytic inverses can be made. I'll brazenly refer to the graph as the Strait of Copeland for the shape of the graph and the existence of the Copeland Islands of Ireland (and, of course, just for fun as doing math should be at its best moments--like surfing after the struggle of paddling out through the waves of a hurricane in the Gulf of Mexico).



The two black curves (the shores of the Strait) are prescribed by

$$y = f(x) = \frac{1}{x} + 1 + x = \frac{1+x+x^2}{x} = \frac{1}{W^{(-1)}(x)} = \frac{1}{W^{(-1)}(\frac{1}{x})}$$

the green line is the quadrant bisector, and the four separated colored curves are strictly monotonically increasing or decreasing sections of the inverse curve $f^{(-1)}(x)$ that are reflections of strictly monotonically increasing or decreasing sections of the shores of the Strait.

Precisely, with f_I, f_{II}, f_{III}, and f_{IV} having the same analytic form $f(x) = x + 1 + \frac{1}{x} = f(\frac{1}{x})$ but different domains and ranges,

I) for $y = f_I(x)$ with the domain $x \geq 1$, the range is $y \geq 3$, and the inverse is ***the right blue curve***

$y = f_I^{(-1)}(x) = \frac{2}{x-1-\sqrt{(1-x)^2-4}} = \frac{x-1+\sqrt{(1-x)^2-4}}{2} = C_B(x) = \frac{1}{C_R(x)}$

with the domain $x \geq 3$ and range $y \geq 1$,

II) for $y = f_{II}(x)$ with the domain $0 < x \leq 1$, the range is $y \geq 3$, and the inverse is ***the right red curve***

$y = f_{II}^{(-1)}(x) = \frac{2}{x-1+\sqrt{(1-x)^2-4}} = \frac{x-1-\sqrt{(1-x)^2-4}}{2} = C_R(x) = \frac{1}{C_B(x)}$

with the domain $x \geq 3$ and range $0 < y \leq 1$,

III) for $y = f_{III}(x)$ with the domain $-1 \leq x < 0$, the range is $y \leq -1$, and the inverse is ***the left blue curve***

$y = f_{III}^{(-1)}(x) = \frac{2}{x-1-\sqrt{(1-x)^2-4}} = \frac{x-1+\sqrt{(1-x)^2-4}}{2} = C_B(x) = \frac{1}{C_R(x)}$

with the domain $x \leq -1$ and range $-1 \leq y < 0$,

IV) for $y = f_{IV}(x)$ with the domain $x \leq -1$, the range is $y \leq -1$, and the inverse is ***the left red curve***

$y = f_{IV}^{(-1)}(x) = \frac{2}{x-1+\sqrt{(1-x)^2-4}} = \frac{x-1-\sqrt{(1-x)^2-4}}{2} = C_R(x) = \frac{1}{C_B(x)}$

with the domain $x \leq -1$ and range $y \leq -1$.

Note the symmetries in the inverse functions induced by $f(x) = f(\frac{1}{x})$.

Stressing once more the incompleteness of the functional reps, the representation of the curve $y = \tilde{f}(x)$ for real x and y as

$y = \tilde{f}(z) = W(x) = \frac{1}{f^{(-1)}(\frac{1}{x})} = \frac{1-z-\sqrt{(1-x)^2-4x^2}}{2x}$

is incomplete, being undefined for domain outside $-1 \leq x \leq \frac{1}{3}$ and range outside $-1 \leq y \leq 1$ in analogy with the rep $y = \sqrt{1-x^2}$ for the upper half of a circle. The full rep for the curve corresponding to the partial rep is

$x = \frac{y}{1+y+y^2} = \frac{1}{y+1+\frac{1}{y}}$ since $y = W^{(-1)}(x) = \frac{x}{1+x+x^2} = \frac{1}{x+1+\frac{1}{x}}$.

Spot checks:

1) $f(1) = z + 1 + \frac{1}{z} \mid_{z=1} = 3$

and

$$f_I^{(-1)}(3) = \frac{2}{z-1-\sqrt{(1-z)^2-4}} \mid_{z=3} = \frac{2}{2-\sqrt{(-2)^2-4}} = 1.$$

2) $f(2) = z + 1 + \frac{1}{z} \mid_{z=2} = \frac{7}{2}$

and

$$f_I^{(-1)}(7/2) = \frac{2}{z-1-\sqrt{(1-z)^2-4}} \mid_{z=7/2} = \frac{2}{\frac{5}{2}-\sqrt{\frac{25}{4}-4}} = 2,$$

3) $f(\frac{1}{2}) = z + 1 + \frac{1}{z} \mid_{z=\frac{1}{2}} = \frac{7}{2}$

but

$$f_I^{(-1)}(7/2) = \frac{2}{z-1-\sqrt{(1-z)^2-4}} \mid_{z=7/2} = \frac{2}{\frac{5}{2}-\sqrt{\frac{25}{4}-4}} = 2.$$

This last inconsistency results in assuming the incorrect inverse of $f(z)$ for $z = \frac{1}{2}$ when, as noted above, it is actually

$$f_{II}^{(-1)}(7/2) = \frac{2}{z-1+\sqrt{(1-z)^2-4}} \mid_{z=7/2} = \frac{2}{\frac{5}{2}+\sqrt{\frac{25}{4}-4}} = 2/4 = 1/2.$$

Direct substitution leads to

$$f_0^{(-1)}(f_0(z)) = \frac{2z}{1+z^2-\sqrt{(1-z^2)^2}} = \frac{2z}{1+z^2-\sqrt{(z^2-1)^2}} = z.$$

The identity

$(W(\frac{1}{z}))^{(-1)} = \frac{1}{W^{(-1)}(z)}$

is obviously satisfied since it trivially means

$W(\frac{1}{z}) \mid_{z \to \frac{1}{W^{(-1)}(z)}} = W(W^{(-1)}(z)) = z.$

Examples of the identity $(W(\frac{1}{z}))^{(-1)} = \frac{1}{W^{(-1)}(z)}$ for

$W(z) = z + b_1 z^2 + b_2 z^3 + \cdots = zB(z) = O_m(z)$:

Example 1)

$W(z) = z \text{ and } W^{(-1)}(z) = z.$

Then

$W(\frac{1}{z}) = \frac{1}{z}$ and

$W(\frac{1}{z}) \mid_{z \to \frac{1}{W^{(-1)}(z)}} = z.$

Example 2)

$W(z) = \frac{z}{1-z} \text{ and } W^{(-1)}(z) = \frac{z}{1+z}.$

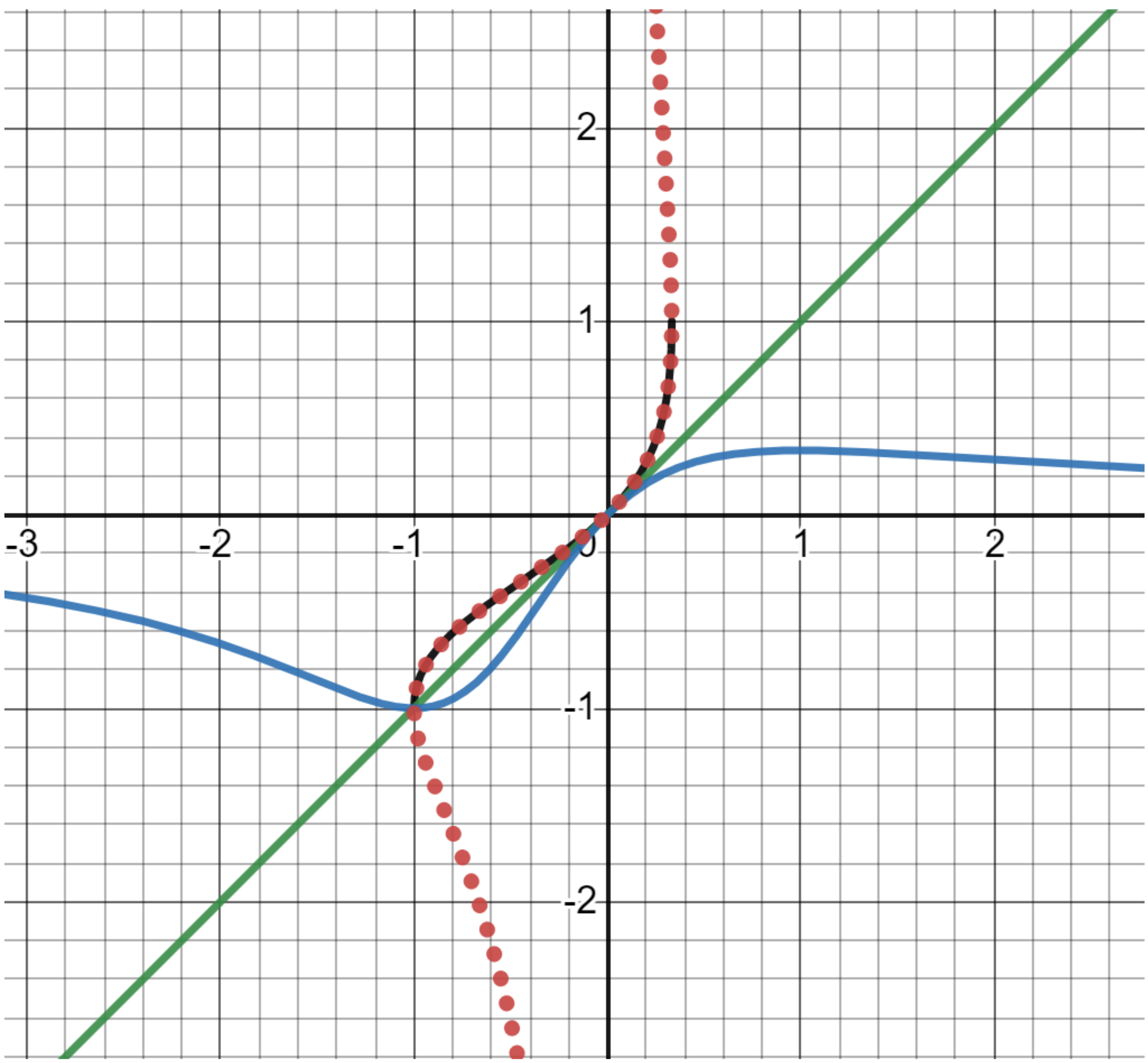
Then

$W(\frac{1}{z}) = \frac{1}{z-1}$ and

$W(\frac{1}{z}) \mid_{z \to \frac{1}{W^{(-1)}(z)}} = \frac{1}{\frac{1+z}{z} - 1} = z.$

Example 3)

Duplicating the first graph above,



switch the curve and its inverse/reflected curve so that now W is the function graphed as the blue curve

$y = W(x) = \frac{x}{1+x+x^2} = \frac{1}{x+1+\frac{1}{x}} = W(\frac{1}{x})$, with the monotonic inverse patchwork

A)

for the right monotonic section of $W(x)$, with $x \geq 1$ and $\frac{1}{3} \geq y > 0$, the inverse is

$$y = W_{+}^{(-1)}(x) = \frac{2x}{1-x-\sqrt{(1-x)^2-4x^2}} = \frac{1-x+\sqrt{(1-x)^2-4x^2}}{2x} = 1/H(x)$$

with domain $0 < x \leq \frac{1}{3}$ and range $y \geq 1$,

B)

for the middle monotonic section of $W(x)$, with $-1 \leq x \leq 1$ and $-1 \leq y \leq \frac{1}{3}$ the inverse is

$$y = W_0^{(-1)}(x) = \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x} = \frac{2x}{1-x+\sqrt{(1-x)^2-4x^2}} = H(x)$$

with the domain $-1 \leq x \leq \frac{1}{3}$ and range $-1 \leq y \leq 1$,

C)

for the left monotonic section of $W(x)$, with $x \leq -1$ and $-1 \leq y < 0$, the inverse is

$$y = W_-^{(-1)}(x) = \frac{2x}{1-x-\sqrt{(1-x)^2-4x^2}} = \frac{1-x+\sqrt{(1-x)^2-4x^2}}{2x} = 1/H(x)$$

with domain $-1 \leq x < 0$ and range $y \leq -1.$

Then again due to the the symmetry in this case

$$W(\tfrac{1}{x}) = W(x) = \frac{x}{1+x+x^2},$$

so we should have

$$W(H(x)) = W(1/H(x)) = x,$$

and we have the simple algebraic check about the origin

$$H(W(x)) = \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x} \Big|_{x \to W(x)} = \frac{1+x^2-\sqrt{(1-x^2)^2}}{2x} = x.$$

To prove Alexandrov's identity again, first note that

$$\left(\frac{1}{f(\frac{1}{z})}\right)^{(-1)} = \frac{1}{f^{(-1)}(\frac{1}{z})} = \frac{1}{f^{(-1)}(z)} \Big|_{z \to \frac{1}{z}}$$

holds since it is equivalent to

$$\frac{1}{f(\frac{1}{z})} \Big|_{z \to \frac{1}{f^{(-1)}(\frac{1}{z})}} = \frac{1}{f(f^{(-1)}(\frac{1}{z}))} = z.$$

This combined with the identities above

$$\frac{O_r(z)}{z^2} = \frac{1}{O_m^{-1}(z)} = \frac{1}{W^{(-1)}(z)} = (\tilde{f}^{(-1)}(z))^{-1} = f(\tfrac{1}{z})$$

imply

$$\tilde{f}(z) = (\frac{1}{f(\frac{1}{z})})^{(-1)} = \frac{1}{f^{(-1)}(\frac{1}{z})}.$$

The identity posed by Alexandrov is

$$\tilde{f}(z) = \frac{1}{f^{(-1)}(\frac{1}{z})}.$$

Let's spot check the validity of this identity using the examples above while collating some of the differing notation related to the separate sets of notes.

Choose

$$\tilde{f}(x) = W(x) = xB(x) = O_m(x) = \frac{x}{1+x+x^2} = \frac{1}{x+1+\frac{1}{x}} = W(\tfrac{1}{x}) = \tilde{f}(\tfrac{1}{x}).$$

The inverse of $y = W_-^{(-1)}(x) = 1/H(x)$ for $-1 \leq x < 0$ and $y \leq -1$ is $y = W(x)$ for $x \leq -1$ and $-1 \leq y < 0$, so for $x \leq -1$, the inverse of $W_-^{(-1)}(\frac{1}{x}) = 1/H(\frac{1}{x})$ is the same section of W.

Then for $x \leq -1$,

$$f(x) = \frac{1}{W^{(-1)}(x)}|_{x\to\frac{1}{x}} = \frac{1}{W^{(-1)}(\frac{1}{x})} = \frac{1}{1/H(\frac{1}{x})} = H(\tfrac{1}{x})$$

$$= \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x}\ |_{x\to\frac{1}{x}} = \frac{2x}{1-x+\sqrt{(1-x)^2-4x^2}}\ |_{x\to\frac{1}{x}}$$

$$= \frac{x-1-\sqrt{(x-1)^2-4}}{2} = \frac{2}{x-1+\sqrt{(x-1)^2-4}},$$

and, from the Strait, the inverse of this function over this domain is

$f^{(-1)}(x) = x + 1 + \frac{1}{x} = f^{(-1)}(\frac{1}{x})$

with domain $x \leq -1 \text{ and range } y \leq -1$,

so

$\frac{1}{f^{(-1)}(\frac{1}{x})} = \frac{1}{x+1+\frac{1}{x}} = \tilde{f}(x),$

as claimed.

Similarly, the inverse of

$y = W_{+}^{(-1)}(x) = \frac{2x}{1-x-\sqrt{(1-x)^2-4x^2}} = \frac{1-x+\sqrt{(1-x)^2-4x^2}}{2x} = 1/H(x)$ for $0 < x \leq \frac{1}{3}$ and $y \geq 1$ is $y = W(x)$ for $x \geq 1$ and $0 < y \leq \frac{1}{3}$, so for $x \geq 3$, the inverse of $W_{+}^{(-1)}(\frac{1}{x}) = 1/H(\frac{1}{x})$ is the same section of W.

Then for $x \geq 3$,

$f(x) = \frac{1}{W^{(-1)}(x)}|_{x\ \frac{1}{x}} = \frac{1}{W^{(-1)}(\frac{1}{x})} = \frac{1}{1/H(\frac{1}{x})} = H(\frac{1}{x})$

$= \frac{1-x-\sqrt{(1-x)^2-4x^2}}{2x} \ |_{x \to \frac{1}{x}} = \frac{2x}{1-x+\sqrt{(1-x)^2-4x^2}} \ |_{x \to \frac{1}{x}}$

$= \frac{x-1-\sqrt{(x-1)^2-4}}{2} = \frac{2}{x-1+\sqrt{(x-1)^2-4}},$

and, from the Strait, the inverse of this function over this domain is

$f^{(-1)}(x) = x + 1 + \frac{1}{x} = f^{(-1)}(\frac{1}{x})$

with domain $x \geq 1$ and range $y \geq 3$,

so again

$\frac{1}{f^{(-1)}(\frac{1}{x})} = \frac{1}{x+1+\frac{1}{x}} = \tilde{f}(x).$

Finally, to account for the two remaining monotonic sections of $f(x) = x + 1 + \frac{1}{x}$, note the inverse of $y = W_0^{(-1)}(x) = H(x)$ for $-1 \leq x \leq \frac{1}{3}$ and $-1 \leq y \leq 1$ is $y = W(x)$ for $-1 \leq x \leq 1$ and $-1 \leq y \leq \frac{1}{3}$, so for both domains $x \leq -1$ and $x \geq 3$, the inverse of $W_0^{(-1)}(\frac{1}{x}) = H(\frac{1}{x})$ is the same section of W.

Then for $x \leq -1$ and $x \geq 3$,

$$f(x) = \frac{1}{W^{(-1)}(x)}|_{x\ \frac{1}{x}} = \frac{1}{W^{(-1)}(\frac{1}{x})} = \frac{1}{H(\frac{1}{x})}$$

$$= \frac{2}{x-1-\sqrt{(x-1)^2-4}} = \frac{x-1+\sqrt{(x-1)^2-4}}{2},$$

and, from the Strait, the inverse of this function is

$$f^{(-1)}(x) = x + 1 + \frac{1}{x} = f^{(-1)}(\frac{1}{x})$$

for $-1 \leq x < 0$ and $y \leq -1$

and for $x \geq 1$ and $y \geq 3$,

giving again

$$\frac{1}{f^{(-1)}(\frac{1}{x})} = \frac{1}{x+1+\frac{1}{x}} = \tilde{f}(x).$$

The complex coordinate change $\omega = z + \frac{1}{z}$ is called the Joukowsky / Joukowski / Zhukovskii / Zhukovsky conformal transformation (function or transform) after Nikolai Zhukovsky, who applied it to aerodynamic analysis of airfoils. It was this revelation that tempted me to strike a similar imperialistic stance and plant my flag, claiming the graph of the real curve $y = x + 1 + \frac{1}{x}$ as the Strait of Copeland. The Zhukovskii function composed with Klein's j-invariant is found on p. 5 of He and Jejjala. See also Suetin p. 36 and 37 with $z + \frac{1}{z}$ expressions.

In a forthcoming set of notes, I'll delve more deeply into many of the operator and associated series identities presented in Alexandrov.

Appendix I: On notation

Throughout these notes I use

$$y = f^{(-1)}(x) = (f(x))^{(-1)}$$

to denote the compositional inverse (in short, the inverse) of a bijective function and

$$y = f^{(-1)}(\tfrac{1}{x}) = (f(u))^{(-1)} \mid_{u=\frac{1}{x}} = (f(x))^{(-1)} \mid_{x \to \frac{1}{x}},$$

which is not typically equal to

$(f(\frac{1}{x}))^{(-1)} = h^{(-1)}(x)$ with $f(\frac{1}{x}) = h(x)$,

whereas I use

$$f^{-1}(x) = (f(x))^{-1} = \tfrac{1}{f(x)}$$

to denote the multiplicative inverse (in short, the reciprocal), for which

$$f^{-1}(\tfrac{1}{x}) = (f(x))^{-1} \mid_{x \to \frac{1}{x}} = (f(\tfrac{1}{x}))^{-1} = \tfrac{1}{f(\frac{1}{x})} = \tfrac{1}{f(x)} \mid_{x \to \frac{1}{x}}.$$

So, pay attention to whether the exponent is -1 or (-1) and be aware that the terms inverse and reciprocal have different meanings in different communities and that the notation varies as well.

To stress again how easy it is to conflate the notation and reasoning, note that it is ***wrong*** to conclude that

$f(\frac{1}{x}) \mid_{x \to \frac{1}{f^{(-1)}(x)}} = f(f^{(-1)}(x)) = x$ implies

$$f^{(-1)}(\tfrac{1}{x}) = \tfrac{1}{f^{(-1)}(x)}.$$

The ***correct inference*** is

$f(\frac{1}{x}) \mid_{x \to \frac{1}{f^{(-1)}(x)}} = f(f^{(-1)}(x)) = x$ implies $(f(\frac{1}{x}))^{(-1)} = \frac{1}{f^{(-1)}(x)}$.

It is true that

$$f^{(-1)}(\frac{1}{x}) = f^{(-1)}(x) \mid_{x \to \frac{1}{x}} = (f(x))^{(-1)} \mid_{x \to \frac{1}{x}}$$

and so

$$f^{(-1)}(\frac{1}{x}) \mid_{x \to \frac{1}{f(x)}} = f^{(-1)}(f(x)) = x.$$

In general composition (or substitution) and compositional inversion do not commute; that is, in general,

$$(f(x))^{(-1)} \mid_{x \to g(x)} \neq (f(x) \mid_{x \to g(x)})^{(-1)} = (f(g(x)))^{(-1)}.$$

For example, for $f(x) = \frac{x}{1+x}$,

$$f^{(-1)}(x) = \frac{x}{1-x},$$

and

$$(f(x))^{(-1)} \mid_{x \to \frac{1}{x}} = (\frac{x}{1-x}) \mid_{x \to \frac{1}{x}} = \frac{1}{x-1}$$

whereas

$$(f(x) \mid_{x \to \frac{1}{x}})^{(-1)} = (f(\frac{1}{x}))^{(-1)} = (\frac{1}{1+x})^{(-1)} = \frac{1}{x} - 1 = \frac{1-x}{x}.$$

Related stuff:

Free probability

"Free probability theory and non-crossing partitions" by Roland Speicher

"Free Probability Theory, Lecture notes, Winter 2018/19" by Roland Speicher

https://video-archive.fields.utoronto.ca/list/event/1804 video lectures by Speicher

"Free Probability Theory and its avatars in representation theory, random matrices, and operator algebras; also featuring: non-commutative distributions" by Roland Speicher

Faber polynomials

"Faber Appells" by myself, posted on my math blog

See "Expansion of analytic functions of an operator in series of Faber polynomials" by Maurice Hasson

"The hitting time subgroup, Łukasiewicz paths and Faber polynomials" by Gi-Sang Cheona, Hana Kima, and Louis W. Shapiro

"Lattice paths and Faber polynomials" by Gessel

"Faber and Grunsky Operators on Bordered Riemann Surfaces of Arbitrary Genus and the Schiffer Isomorphism" by Mohammad Shirazi

"Series of Faber Polynomials" by Suetin