Conjugate Appells for the Catalans

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First set of conjugates with the parent function $B(t) = t - t^2$

Using the notation and identities of the my previous two blog posts on conjugation and umbral calculus, let the parent function and its compositional inverse be

$$B(t) = t - t^2,$$

and

$$B^{(-1)}(t) = \frac{1-\sqrt{1-4t}}{2}.$$

Then the e.g.f.s for the moments of the associated Appell sequence and its conjugate are

$$A(t) = \frac{t}{B(t)} = \frac{1}{1-t} = e^{a.t} = 1 + t + 2!\frac{t^2}{2!} + \cdots$$

and

$$\bar{A}(t) = \frac{B^{(-1)}(t)}{t} = \frac{1-\sqrt{1-4t}}{2t} = e^{\bar{a}.t}$$

$$= 1 + t + 2!\; Cat_2 \frac{t^2}{2!} + 2!\; Cat_3 \frac{t^3}{3!} + \cdots .$$

The conjugate set of moments are the factorials $a_n = n!$ and the quadruple factorials (cf. A001813) $\bar{a}_n = (2n)!/n! = n!\; Cat_n$, where Cat_n are the Catalan numbers of A000108, $[Cat] = [1, 1, 2, 5, 14, 42, ...]^T$. (These are also simply related to the perfect matchings of the vertices of the n-simplices / hypertriangles / hypertetrahedrons.)

The binomial Sheffer polynomial sequence associated with the parent function has the e.g.f.

$$e^{B.(x)t} = e^{xB(t)} = e^{x(t-t^2)}$$

with the coefficient matrix $[B] =$ padded A119275 (by padded I mean an extra initial row and column of all zeros except $B_{0,0} = 1$ as is true for all canonical binomial Sheffer sequences).

The umbral inverse binomial Sheffer polynomial sequence has the e.g.f.

$$e^{\hat{B}.(x)t} = e^{x\hat{B}(t)} = e^{xB^{(-1)}(t)} = \exp(x(\frac{1-\sqrt{1-4t}}{2t}))$$

and the coefficient matrix $[\hat{B}] = [B^{(-1)}] = [B]^{-1} =$ padded A119274.

The associated Appell polynomial sequence has the e.g.f.

$$e^{A.(x)t} = A(t)\ e^{xt} = \frac{1}{1-t}\ e^{x}t = e^{a.t}\ e^{xt},$$

and the coefficient matrix for the row polynomials $A_n(x) = (a. + x)^n$ is $[A] =$ A094587.

The conjugate Appell polynomial sequence has the e.g.f.

$$e^{\bar{A}.(x)t} = \bar{A}(t)\ e^{xt} = \frac{1-\sqrt{1-4t}}{2t}\ e^{x}t = e^{\bar{a}.t}\ e^{xt}.$$

The coefficient matrix for the row polynomials $\bar{A}_n(x) = (\bar{a}. + x)^n$ is not in the OEIS.

The conjugate identities discussed in the two previous posts and in the MathOverflow question "Combinatorial proof of a matrix equation" lead to a formula for self-convolutions of the Catalan numbers.

The e.g.f. of the binomial convolution of two moment sequences is

$$e^{a1.t}e^{a2.t} = e^{(a1.+a2.)t} = e^{bc.t}$$

with the binomial convolution explicitly (and laboriously, for the sigma purists) given by

$$(bc.)^n = bc_n = (a1. + a2.)^n = \sum_{k=0}^{n}\ \binom{n}{k}\ a1_k\ a2_{n-k},$$

and the convolution of the two sequences $a1_n = n!\ Na1_n$ and $a2_n = n!\ Na2_n$ has the o.g.f.

$$\frac{1}{1-Nc.t} = \frac{1}{1-Na1.t}\ \frac{1}{1-Na2.t}$$

with

$$Nc_n = \sum_{k=0}^{n}\ \ Na1_k\ Na2_{n-k},$$

corresponding to the multiplication of o.g.f.s.

Note the potential pitfall in interpreting self-multiplication of an e.g.f. as

$$A^2(t) = A(t)A(t) = e^{a.t}e^{a.t} = e^{(a.+a.)t} = e^{2a.t}.$$

THIS IS FALSE. The umbral evaluation should come before multiplying the umbral rep of the e.g.f. by itself, not after; this is,

$$A(t)A(t) = (e^{a.D_{x=0}}e^{xt}) \cdot (e^{a.D_{x=0}}e^{xt})$$

$$\neq e^{a.D_{x=0}}e^{xt}e^{xt} = e^{a.D_{x=0}}e^{2xt} = e^{2a.t}.$$

However, in this case, treating the same umbral character as two distinct umbral characters $a1.$ and $a2.$, reducing the equation to an identity between analytic series, next lowering the exponents $(a1.)^m = a1_m$ and $(a2.)^m = a2_m$, and then finally letting $a1_k = a2_k = a_k$ (this process) gives the correct result. Following this last procedure, self-convolution explicitly gives

$$A^2(t) = A(t)A(t) = (e^{a.D_{x=0}}e^{xt}) \cdot (e^{a.D_{x=0}}e^{xt}) = e^{(a1.+a2.)t}$$

$$= \sum_{n\geq 0}(a1. + a2.)^n \frac{t^n}{n!} = \sum_{n\geq 0}(\sum_{k=0}^{n} \binom{n}{k} \, a1_k \, a2_{n-k}) \, \frac{t^n}{n!}$$

$$= \sum_{n\geq 0}(\sum_{k=0}^{n} \binom{n}{k} \, a_k \, a_{n-k}) \, \frac{t^n}{n!}$$

$$= \sum_{n\geq 0}(\sum_{k=0}^{n} \, \frac{a_k}{k!} \, \frac{a_{n-k}}{(n-k)!}) \, t^n \, .$$

Consequently, the e.g.f. of m self-convolutions of the Catalan numbers is given by

$$MC(t) = e^{mc.t} = (\bar{A}(t))^{m+1} = (\frac{1-\sqrt{1-4t}}{2t})^{m+1},$$

where $mc.$ is a single umbral character for the Taylor series coefficients of $MC(t)$ and $(mc.)^k = mc_k \neq m \cdot c_k$, but this is just the moment e.g.f. of the Appell sequence e.g.f.

$$e^{MC.(x)t} = e^{(mc.+x)t} = MC(t) \; e^{xt} = (\frac{1-\sqrt{1-4t}}{2t})^{m+1} \; e^{xt},$$

which has the coefficient matrix $[MC] = [\bar{A}]^{m+1}$ with the first column $[mc]$ having the elements

$$mc_n = MC_n(0) = \frac{Cat_n^{(m)}}{.n!}$$

with $Cat_n^{(m)}$ the n'th element of the m'th self-convolution of the Catalan numbers Cat_n.

Then the conjugation relation

$$[\bar{A}]^{m+1} = [B]^{-1}[A]^{m+1}[B]$$

leads to

$$[mc] = [B]^{-1}[a^{(m+1)}],$$

where $[a^{(m+1)}]$ has the e.g.f.

$$e^{a.^{(m+1)}t} = \frac{1}{(1-x)^{m+1}} = (1-x)^{-(m+1)},$$

so

$$a_n^{(m+1)} = \frac{(m+n)!}{m!} = (m+n)\cdots(m+1),$$

and

$$Cat_n^{(m)} = \frac{mc_n}{n!}.$$

Numerical checks for the first few rows of the matrix equation agree with expansions of the associated e.g.f.s and the corresponding OEIS entries, giving for the first self-convolution the shifted Catalan numbers with the initial one removed and, for the second, shifted A000245 with the first zero removed.

As an illustration and sanity check;

from the e.g.f. $e^{\hat{B}.(x)t} = e^{(\frac{1-\sqrt{1-4t}}{2})t}$

$[B^{(-1)}] = [B]^{-1} = [\hat{B}] \rightarrow$

$$\begin{bmatrix} 1 \\ 0 & 1 \\ 0 & 2 & 1 \\ 0 & 12 & 6 & 1 \\ 0 & 120 & 60 & 12 & 1 \end{bmatrix},$$

$[a^{(3)}] \rightarrow$

$$\begin{bmatrix} 1 = 1 \\ 3 = 3 \\ 12 = 6 \cdot 2! \\ 60 = 10 \cdot 3! \\ 360 = 15 \cdot 4! \end{bmatrix},$$

and

$[B]^{-1}[\bar{a}^{(3)}] = [\bar{a}^{(3)}] \rightarrow$

$$\begin{bmatrix} 1 = 1 \\ 3 = 3 \\ 18 = 9 \cdot 2! \\ 168 = 28 \cdot 3! \\ 2160 = 90 \cdot 4! \end{bmatrix},$$

with $a_n^{(3)} = \frac{(n+2)!}{2!}$, the Taylor series coefficients for $(1-t)^{-3}$.

The associated e.g.f.s give

$$(\frac{1-\sqrt{1-4t}}{2t})^3 = 1 + 3t + 9t^2 + 28t^3 + 90t^4 + \cdots$$

$= 1 + 3t + 18\frac{t^2}{2!} + 168\frac{t^3}{3!} + 2160\frac{t^4}{4!} + \cdots$

(cf. A000245)

and

$(1-t)^{-3} = 1 + 3t + 6t^2 + 10t^3 + 15t^4 + \cdots$

$= 1 + 3t + 12\frac{t^2}{2!} + 60\frac{t^3}{3!} + 360\frac{t^4}{4!} + \cdots ,$

giving the third diagonal and column of the lower triangular Pascal matrix A007318.

This all agrees with the functional composition rep

$\bar{A}(t) = A(\hat{B}(t)) = A(B^{(-1)}(t))$

$= \frac{B^{(-1)}(t)}{t} = \frac{t}{B(t)} \mid_{t=B^{(-1)}(t)} = A(t) \mid_{t=B^{(-1)}(t)},$

so

$MC(t) = (\bar{A}(t))^{m+1} = A^{m+1}(\hat{B}(t)) = A^{m+1}(B^{(-1)}(t))$

$= (\frac{B^{(-1)}(t)}{t})^{m+1} = (\frac{t}{B(t)})^{m+1} \mid_{t=B^{(-1)}(t)} = A^{m+1}(t) \mid_{t=B^{(-1)}(t)},$

and this has the matrix conjugation rep

$[MC] = [B]^{-1}[A]^{m+1}[B].$

Second set of conjugates with the parent function $B(t) = \frac{t}{1+t}$

I'm now going to introduce a new parent function and another conjugation relation involving also the permutation matrix A094587, so I need to relabel the matrices above.

Above ***the first parent function***

$B(t) = t - t^2 = BC^{(-1)}(t)$

with the inverse

$$B^{(-1)}(t) = \frac{1-\sqrt{1-4t}}{2} = BC(t),$$

related to the Catalan numbers A000108,

is used to relate the Appell Permutation matrix

$[A] = [AP]$, A094587,

associated with the Appell moment e.g.f.

$$A(t) = \frac{t}{B(t)} = \frac{1}{1-t} = AP(t)$$

to the Appell Catalan matrix

$$[\bar{A}] = [AC]$$

associated with the conjugate Appell moment e.g.f.

$$\bar{A}(t) = \frac{B^{(-1)}(t)}{t} = \frac{1-\sqrt{1-4t}}{2t} = AC(t)$$

as a conjugate pair via conjugation with the Binomial Catalan matrix

$[B^{(-1)}] = [BC]$, padded A119274;

that is, the conjugation relation

$$[AP] = [BC]^{-1}[AC][BC]$$

Is derived above.

Several years ago, I established also another conjugation relation using
the second parent function

$$B(t) = \frac{t}{1+t} = BL^{(-1)}(t),$$

associated with the signed Lah polynomials, or normalized Laguerre polynomials of order -1,

with the inverse

$B^{(-1)}(t) = \frac{t}{1-t} = -B(-t) = BL(t)$.

With this new parent function, we have the new associations

$\bar{A}(t) = \frac{B^{(-1)}(t)}{t} = \frac{1}{1-t} = AP(t)$

and

$A(t) = \frac{t}{B(t)} = 1 + t = AT(t)$,

for which AT is the 'acronym' for ***A**ppell 1+**T*** serving as a mnemonic for suggesting only the first two moments--the constant term and the coefficient for the linear term $t = T$--are nonzero for this Appell sequence. Then

$[AP]$,

and

$[AT]$, unsigned A132013 ,

are a conjugate pair with respect to the binomial Lah matrix

$[BL]$, unsigned A111596

with the e.g.f.

$e^{B^{(-1)}(t)x} = e^{-B(-t)x} = e^{\frac{t}{1-t}x} = e^{BL.(x)t} = e^{BL(t)x}$.

Note that

$[BL]^{-1}$ is a signed version of $[BL]$

with e.g.f.

$e^{B(t)x} = e^{\frac{t}{1+t}x} = e^{-B^{(-1)}(-t)x} = e^{-BL(-t)x} = e^{-BL.(-x)t}$,

so the row polynomials of

$[BL]^{-1} = [\widehat{BL}]$

are

$$\widehat{BL}_n(x) = (-1)^n \; BL_n(-x).$$

Then the conjugation relation

$$[\bar{A}] = [\hat{B}][A][B] = [B]^{-1}[A][B]$$

becomes for the new parent function ***the new conjugation relation***

$$[AP] = [BL][AT][BL]^{-1}.$$

Consequently,

$$[BC]^{-1}[AC][BC] = [BL][AT][BL]^{-1}$$

and

$$[AC] = [BC][BL][AT][BL]^{-1}[BC]^{-1}$$

or

$$[AT] = [BL]^{-1}[BC]^{-1}[AC][BC][BL].$$

The product of coefficient matrices for any two Sheffer sequences, $[S12] = [S1][S2]$, corresponds to the umbral composition $S12_n(x) = S1_n(S2.(x))$. For two binomial Sheffer sequences, we have

$$B12_n(x) = B1_n(B2.(x))$$

and

$$e^{xB12(t)} = e^{B12.(x)t} = e^{B1.(B2.(x))t} = e^{B2.(x)B1(t)} = e^{xB2(B1(t))},$$

so

$$B12(t) = B2(B1(t)).$$

Consequently, the binomial coefficient matrix

$$[BCL] = [BC][BL]$$

has the e.g.f.

$$e^{BCL.(x)t} = e^{BCL(t)x}$$

with

$$BCL(t) = BL(BC(t)) = \frac{BC(t)}{1-BC(t)} = \sum_{n\geq 1}(BC(t))^n$$

$$= \frac{\frac{1-\sqrt{1-4t}}{2}}{1-\frac{1-\sqrt{1-4t}}{2}} = \frac{1-\sqrt{1-4t}}{1+\sqrt{1-4t}} = t(\frac{1-\sqrt{1-4t}}{2t})^2$$

$$= \frac{1-\sqrt{1-4t}}{2t} - 1$$

$$= t + 2t^2 + 5t^3 + 14t^4 + 42t^5 + 132t^6 + \cdots ,$$

which is the o.g.f. of the Catalan numbers minus one,

with inverse

$$BCL^{(-1)}(t) = BC^{(-1)}(BL^{-1}(t))$$

$$= BC^{(-1)}(-BL(-t)) = (-BL(-t)) - (-BL(-t))^2$$

$$= \frac{t}{1+t} - \frac{t^2}{(1+t)^2} = \frac{t}{(1+t)^2}$$

with the associated matrix rep

$$[BCL^{(-1)}] = [BCL]^{-1} = [BL]^{-1}[BC]^{-1}.$$

Reducing

$$[AT] = [BL]^{-1}[BC]^{-1}[AC][BC][BL]$$

gives

$$[at] = [BL]^{-1}[BC]^{-1}[ac] = [BCL]^{-1}[ac].$$

Now for a sanity check with numerics:

From the e.g.f. $e^{\widehat{BCL}.(x)t} = e^{\frac{t}{(1+t)^2}x}$,

$$[BCL]^{-1} = [\widehat{BCL}] \rightarrow \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & -4 & 1 & & \\ 0 & 18 & -12 & 1 & \\ 0 & -96 & 120 & -24 & 1 \end{bmatrix},$$

$$[ac] \rightarrow \begin{bmatrix} 1 = 1 \\ 1 = 1 \\ 4 = 2 \cdot 2! \\ 30 = 5 \cdot 3! \\ 336 = 14 \cdot 4! \end{bmatrix},$$

i.e., A001761,

and

$$[at] \rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

which agree numerically with calculations from the reduced matrix equation.

For another spot check, from the e.g.f. $e^{BCL.(x)t} = \exp((\frac{1-\sqrt{1-4t}}{2t} - 1)x)$,

$$[BCL] \rightarrow \begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 4 & 1 & & \\ 0 & 30 & 12 & 1 & \\ 0 & 336 & 168 & 24 & 1 \end{bmatrix},$$

which agrees numerically as the inverse of the submatrix for $[BCL]^{-1}$ above and is consistent with

$$[ac] = [BCL][at].$$

.
For easy reference:

from the e.g.f. $e^{BC.(x)t} = e^{(\frac{1-\sqrt{1-4t}}{2})x}$ for padded A119274,

$[BC] =\rightarrow$

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 2 & 1 & & \\ 0 & 12 & 6 & 1 & \\ 0 & 120 & 60 & 12 & 1 \end{bmatrix},$$

from the e.g.f. $e^{\widehat{BC}.(x)t = e^{(t-t^2)x}}$ for padded A119275,

$[BC]^{-1} = [\widehat{BC}] \rightarrow$

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & -2 & 1 & & \\ 0 & 0 & -6 & 1 & \\ 0 & 0 & 12 & -12 & 1 \end{bmatrix},$$

from the e.g.f. $e^{BL.(x)t} = e^{(\frac{t}{1-t})x}$ for unsigned A111596,

$[BL] \rightarrow$

$$\begin{bmatrix} 1 & & & & \\ 0 & 1 & & & \\ 0 & 2 & 1 & & \\ 0 & 6 & 6 & 1 & \\ 0 & 24 & 36 & 12 & 1 \end{bmatrix},$$

from the e.g.f. $e^{\widehat{BL}.(x)t} = e^{(\frac{t}{1+t})x}$ for A111596,

$[BL]^{-1} = [\widehat{BL}] \rightarrow$

$$\begin{bmatrix} 1 \\ 0 & 1 \\ 0 & -2 & 1 \\ 0 & 6 & -6 & 1 \\ 0 & -24 & 36 & -12 & 1 \end{bmatrix},$$

from the e.g.f. $e^{AC.(x)t} = \frac{1-\sqrt{1-4t}}{2t}\ e^{xt}$,

$[AC] \rightarrow$

$$\begin{bmatrix} 1 \\ 1 & 1 \\ 4 = 2 \cdot 2! & 2 & 1 \\ 30 = 5 \cdot 3! & 12 & 3 & 1 \\ 336 = 14 \cdot 4! & 120 & 24 & 4 & 1 \end{bmatrix},$$

with the first column of normalized Catalan numbers,

from the e.g.f. $e^{AT.(x)t} = (1+t)\ e^{xt}$ for unsigned A132013,

$[AT] \rightarrow$

$$\begin{bmatrix} 1 \\ 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 4 & 1 \end{bmatrix},$$

and, from the e.g.f. $e^{AP.(x)t} = \frac{1}{1-t}\ e^{xt}$ for A094587, the permutation matrix,

$[AP] \rightarrow$

$$\begin{bmatrix} 1 \\ 1 & 1 \\ 2 & 2 & 1 \\ 6 & 6 & 3 & 1 \\ 24 & 24 & 12 & 4 & 1 \end{bmatrix}.$$

Note that $[AT]$ with the first subdiagonal negated is the inverse matrix of the Appell permutation matrix $[AP]$.