# Generators, Inversion, and Matrix, Binomial, and Integral Transforms

Tom Copeland, tcjpn@msn.com

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# Part I

### The Borel-Laplace transform of generating functions

The Borel-Laplace transform formally converts an exponential generating function (e.g.f.) for a sequence into its corresponding ordinary generating function (o.g.f.): The Borel-Laplace transform of the e.g.f.

$$f_E(t) = \sum_{n\geq 0} a_n \frac{t^n}{n!} = \sum_{n\geq 0} \frac{(a.t)^n}{n!} = e^{a.t}$$

gives the o.g.f.

$$f_O(t) = \sum_{n\geq 0} a_n\, t^n = \sum_{n\geq 0} (-1)^n \binom{-1}{n} a_n\, t^n = \sum_{n\geq 0} (-1)^n \binom{-1}{n} (a,t)^n = 1/(1-a.t)\,;$$

that is,

$$L_B[f_E(t)] = \int_0^\infty \frac{1}{z} e^{-\frac{t}{z}} f_E(t)\, dt = \sum_{n\geq 0} a_n \int_0^\infty \frac{1}{z} e^{-\frac{t}{z}} \frac{t^n}{n!}\, dt = \sum_{n\geq 0} a_n\, z^n = f_O(z)\,,$$

or, more elegantly,

$$L_B[f_E(t)] = \int_0^\infty \frac{1}{z}\, e^{-\frac{t}{z}} e^{a.t}\, dt = \int_0^\infty \frac{1}{z}\, e^{-(\frac{1}{z}-a.)t}\, dt = \frac{1}{z} \frac{1}{\frac{1}{z}-a.} = \frac{1}{1-a.z} = f_O(z)\,.$$

With a change of the variable for integration from $t$ to $zt$, the transform appears as

$$L_B[f_E(t)] = \int_0^\infty e^{-t} f_E(zt)\, dt = \int_0^\infty e^{-t} e^{a.zt}\, dt$$
$$= \int_0^\infty e^{-(1-a.z)t}\, dt = \frac{1}{1-a.z} = c_{-1}$$

with the binomial transform (more on this transform in the Appendix)

$$(1-a.z)^n = \sum_{k=0}^{n} (-1)^n \binom{n}{k} (a.z)^n = \sum_{k=0}^{n} (-1)^n \binom{n}{k} a_n z^n = (c.)^n = c_n \,,$$

naturally extended to

$$(1-a.z)^{-1} = c_{-1} = 1/(1-a.z) = f_O(z)\,.$$

# Part II

### Binomial Sheffer polynomials, their reverts, and a Laplace transform

A binomial Sheffer sequence of polynomials $S_n(z)$ can be defined by its e.g.f.

$$e^{t\,S.(z)} = e^{z\,G(t)},$$

where $S_0(z) = 1$ and $G$ has a compositional inverse with $G(0) = 0$, and satisfy, among other interesting relations,

$$[S.(x) + S.(y)]^n = \sum_{k=0}^{n} \binom{n}{k} (S.(x))^{n-k}(S.(y))^k = \sum_{k=0}^{n} \binom{n}{k} S_{n-k}(x)S_k(y) = S_n(x+y)\,.$$

The reverse polynomials are

$$R_n(z) = z^n S_n(1/z)$$

and satisfy

$$e^{t\,R.(z)} = e^{tz\,S.(1/z)} = e^{\frac{1}{z}G(tz)}.$$

An example of particular importance to the ultimate objective of these notes is the sequence of Stirling polynomials of the first kind (OEIS-A008275, the falling factorials, and their reverses A008276) with $G(t) = ln(1+t)$:

$$e^{tS.(z)} = e^{z\,ln(1+t)} = (1+t)^z = \sum_{n\geq 0} \binom{z}{n} t^n = \sum_{n\geq 0} (z)_n \frac{t^n}{n!}$$

with the first few being

$$S_0(z) = 1$$

$$S_1(z) = z$$

$$S_2(z) = z(z-1) = z^2 - z$$

$$S_3(z) = z(z-1)(z-2) = z^3 - 3z + 2$$

$$S_4(z) = z^4 - 6z^3 + 11z^2 - 6z\,.$$

The reverse polynomials satisfy

$$e^{tR.(z)} = e^{t\frac{1}{z}S.(z)} = e^{\frac{1}{z}G(tz)} = e^{\frac{1}{z}ln(1+tz)} = (1+tz)^{\frac{1}{z}} \sum_{n\geq 0} \binom{1/z}{n} (zt)^n$$

with the first few being

$$R_0(z) = 1$$

$$R_1(z) = 1$$

$$R_2(z) = 1 - z$$

$$R_3(z) = 1 - 3z + 2z^2$$

$$R_4(z) = 1 - 6z + 11z^2 - 6z^3\,.$$

Representing the coefficients of the polynomials as lower triangular matrices,

$$S = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 \\ 0 & -6 & 11 & 6 & 1 \end{pmatrix}$$

$$R = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ 1 & -6 & 11 & -6 & 0 \end{pmatrix}.$$

Note that shifted columns of $R$ are the diagonals of $S$ , and, conversely, the shifted columns of $S$ are the diagonals of $R$, a relation that holds for all pairs of reverse matrices.

The formal transform

$$\int_0^\infty e^{-\omega} e^{-[\omega u - \frac{1}{z}G(u\omega z)]} d\omega = \int_0^\infty e^{-\omega} e^{-\omega u}\, e^{\frac{1}{z}G(u\omega z)} d\omega$$

$$= \int_0^\infty e^{-\omega} e^{-\omega u}\, e^{u\omega z S.(1/z)} d\omega = \int_0^\infty e^{-\omega} e^{-\omega u}\, e^{u\omega R.(z)} d\omega$$

$$= \int_0^\infty e^{-\omega} e^{-\omega u(1-R.(z))} d\omega = \int_0^\infty e^{-\omega(1+(1-R.(z))u)} d\omega$$

$$= \frac{1}{1+(1-R.(z))u} = \sum_{n\geq 0} (1-R.(z))^n (-u)^n,$$

gives the o.g.f. with negated argument of the binomial transform of the reverse polynomials. Changing the integration variable gives

$$\int_0^\infty \frac{1}{z}\, e^{-\frac{1}{z}\omega} e^{-\frac{1}{z}[\omega u - G(u\omega)]} d\omega = \int_0^\infty \frac{1}{z}\, e^{-\frac{1}{z}[\omega + \omega u - G(u\omega)]} d\omega =$$

$$= \int_0^\infty \frac{1}{z}\, e^{-\frac{1}{z}\omega[1+(1-R.(z))u]} d\omega = \sum_{n\geq 0} (1 - R.(z))^n (-u)^n .$$

# Part III

### Compositional inversion and one-forms for the Borel-Laplace transform

Given a pair of functions mutually inverse under composition

$$f(\omega) = \sigma \;\text{ and }\; f^{-1}(\sigma) = \omega\,,$$

the inverse function theorem follows from the chain rule applied to the differentials:

$$d\sigma = df(\omega) = f^{'}(\omega)\, d\omega = f^{'}(\omega)\, df^{-1}(\sigma) = f^{'}(\omega)[f^{-1}(\sigma)]^{'} d\sigma\,,$$

so

$$[f^{-1}(\sigma)]^{'} = 1/f^{'}(\omega)$$

for $(\sigma, \omega)$ satisfying the inverse relations. Then also

$$-de^{-\frac{1}{z}f(\omega)} = -de^{-\frac{1}{z}\sigma}$$

$$f^{'}(\omega) e^{-\frac{1}{z}f(\omega)} d\omega = e^{-\frac{1}{z}\sigma} d\sigma\,,$$

and

$$\int e^{-\frac{1}{z}f(\omega)} d\omega = \int \frac{1}{f'(\omega)} e^{-\frac{1}{z}\sigma} d\sigma = \int e^{-\frac{1}{z}\sigma} [f^{-1}(\sigma)]^{'} d\sigma\,.$$

For suitable inverse pairs, the Borel-Laplace transform is then

$$\int_0^\infty \frac{1}{z} e^{-\frac{1}{z}f(\omega)} d\omega = \int_0^\infty \frac{1}{z} e^{-\frac{1}{z}\sigma} [f^{-1}(\sigma)]^{'} d\sigma\,.$$

# Part IV

### Compositional inversion of -ln(1-b.x) and associated infinigen

The partition polynomials $P_n(b_1, .., b_{n)})$ of A133932 provide a means of generating the Taylor series order by order for the compostional inverse of an invertible function represented as a logarithmic series, i.e., as

$$f(x) = -ln(1 - b.x) = \sum_{n\geq 1} b_n \frac{x^n}{n!}.$$

They can be produced by an infinitesimal generator (infinigen)

$$g(x)\,\frac{d}{dx} = \frac{d}{df(x)} = \frac{1}{f'(x)}\frac{d}{dx}$$

since

$$exp\left[t\,\frac{d}{df(\omega)}\right]\omega = exp\left[t\,\frac{d}{d\sigma}\right] f^{-1}(\sigma) = f^{-1}(t+\sigma) = f^{-1}[t + f(\omega)],$$

so

$$exp\left[t\,g(x)\frac{d}{dx}\right] x\,|_{x=0} = f^{-1}(t)$$

(see A145271 for more info). Then the infinigen is

$$g(x)\frac{d}{dx} = \frac{d}{df(x)} = \frac{1}{b_1 + b_2 x + b_3 x^2 + ...}\,\frac{d}{dx}\,,$$

and the partition polynomials are generated by iterated derivation by the infinigen

$$\left(g(x)\frac{d}{dx}\right)^n x\,|_{x=0} = \left(\frac{1}{b_1 + b_2 x + b_3 x^2 + ...}\,\frac{d}{dx}\right)^n x\,|_{x=0} = P_n(b_1, b_2, ..., b_n).$$

The first few partition polynomials are

$$P_1(b_1) = \frac{1}{b_1}\,[1]$$

$$P_2(b_1, b_2) = \frac{1}{b_1^3}\,[-b_2]$$

$$P_3(b_1, b_2, b_3) = \frac{1}{b_1^5}\,[3b_2^2 - 2b_1 b_3]$$

$$P_4(b_1, b_2, b_3, b_4) = \frac{1}{b_1^7}\,[-15b_2^3 + 20b_1 b_2 b_3 - 6b_1^2 b_4]$$

$$P_5(b_1, b_2, b_3, b_4 b_5) = \frac{1}{b_1^9}\,[105b_2^4 - 210b_1 b_2^2 b_3 + 90b_1^2 b_2 b_4 + 40b_1^2 b_3^2 - 24b_1^3 b_5]\,.$$

The partition polynomials apparently reduce to shifted reverse polynomials of A111999 with $b_1 = 1$ and $b_n = (-u)^n$ for $n > 1$, i.e., with

$$f(x) = -ln(1 - b.x) = (1 + u)x - ln(1 + ux)\,.$$

In the next section, it is shown that A133932 is indeed a refinement of A111999 by this choice of $f(x)$ in the Borel-Laplace rep for compositional inversion.

The infinigen is related to an aerated refinement of the Pascal triangle which in turn is related to certain Fibonnaci polynomials:

$$g(x)\frac{d}{dx} = \frac{1}{b_1 + b_2 x + b_3 x^2 + ...}\,\frac{d}{dx}$$

$$\left[\sum_{n\geq 0} D_n(b_1, ..., b_n, b_{n+1})\,x^n\right]\frac{d}{dx}$$

with the first few partition polynomials for the infinigen as

$$D_0(b_1) = \frac{1}{b_1}\,[1]$$

$$D_1(b_1, b_2) = \frac{1}{b_1^2}\,[-b_2]$$

$$D_2(b_1, b_2, b_3) = \frac{1}{b_1^3}\,[b_2^2 - b_1 b_3]$$

$$D_3(b_1, b_2, b_3, b_4) = \frac{1}{b_1^4}\,[-b_2^3 + 2b_1 b_2 b_3 - b_1^2 b_4]$$

$$D_4(b_1, ..., b_5) = \frac{1}{b_1^5}\,[b_2^4 - 3b_1 b_2^2 b_3 + b_1^2(b_3^2 + 2b_2 b_4) - b_1^3 b_5]$$

$$D_5(b_1, ..., b_6) = \frac{1}{b_1^6}\,[-b_2^5 + 4b_1 b_2^3 b_3 - 3b_1^2(b_2 b_3^2 + b_2^2 b_4) + 2b_1^3(b_2 b_5 + b_3 b_4) - b_1^4 b_6].$$

In the next part, the partition polynomials in square backets for the infinigen are shown to reduce to shifted, signed rows of the Pascal matrix for the special case

$$f(x) = \omega + u\omega - ln(1 + u\omega) \quad \text{with} \quad \mathrm{b_1} = 1 \text{ and } \mathrm{b_n} = (-\mathrm{u})^{\mathrm{n}} \text{ otherwise.}$$

# Part V

Reduced partition polynomials (A111999) and their associated infinigen

Let's weave together the maneuvers introduced above to show that A111999 is a presentation of the reduced partition polynomials of A133932. Let

$$f(\omega) = h(\omega; u) = \omega + u\omega - ln(1 + u\omega) = \omega + \sum_{n \geq 2} (-u)^n \frac{\omega^n}{n}.$$

Then

$$\int_0^\infty \frac{1}{z} e^{-\frac{1}{z}\sigma} [h^{-1}(\sigma; u)]' \, d\sigma = \int_0^\infty \frac{1}{z} e^{-\frac{1}{z} h(\omega;u)} d\omega$$

$$= \int_0^\infty \frac{1}{z} e^{-\frac{1}{z}[\omega + u\omega - ln(1+u\omega)]} d\omega = \int_0^\infty e^{-[\omega + u\omega - \frac{1}{z} ln(1+u\omega z)]} d\omega$$

$$= \int_0^\infty e^{-[\omega + u\omega]} (1 + u\omega z)^{\frac{1}{z}} d\omega = \int_0^\infty e^{-[\omega+u\omega]} e^{u\omega R.(z)} d\omega$$

$$= \int_0^\infty e^{-\omega} e^{-u\omega(1-R.(z))} d\omega = \sum_{n \geq 0} (-1)^n (1 - R.(z))^n u^n,$$

so the integral transform evaluates to the o.g.f. of the binomial transforms of the reversed row polynomials given above of the Stirling polynomials of the first kind. The row polynomials in the variable $u$ of the inverse function's e.g.f. are the coefficients of this o.g.f. as a power series in $z$. Since

$$\sum_{n \geq 0} (-1)^n (1 - R.(z))^n u^n = \sum_{n \geq 0} (-1)^n u^n \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j \geq 0} R_{k,j} z^j$$

$$= \sum_{j \geq 0} z^j \sum_{n \geq 0} (-1)^n u^n \sum_{k \geq 0}^{n} (-1)^k \binom{n}{k} R_{k.j},$$

the n-th row polynomial (Taylor series coeficient) of the inverse's e.g.f. for $n \geq 1$ is

$$P_n(u) = \sum_{m \geq 0} P_{n,m} u^m = \sum_{m \geq 0} (-1)^m u^m \sum_{k \geq 0}^{m} (-1)^k \binom{m}{k} R_{k,n-1},$$

so the coefficient of $u^m$ of the polynomial is a binomial transform of the columns of $R$ and, therefore, the diagonals of $S$. For example,

$$P_{3,4} = (-1)^4 \sum_{k \geq 0}^{4} (-1)^k \binom{4}{k} R_{k,2}$$

equals the sum of the dot product terms along a column of $R$

$$\begin{pmatrix} 1 & 0 & 1\cdot 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -4\cdot 0 & 0 & & 0 & 0 & 0 \\ 1 & -1 & 6\cdot 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & -4\cdot 2 & 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 1\cdot 11 & -6 & 0 & 0 & 0 & 0 \\ 1 & -10 & 35 & -50 & 24 & 0 & 0 & 0 \\ 1 & -15 & 85 & -225 & 274 & -120 & 0 & \\ 1 & -21 & 175 & -735 & 1624 & -1764 & 720 & 0 \end{pmatrix}$$

equals the sum of the dot product terms along a diagonal of $S$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 6\cdot 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -4\cdot 2 & -3 & 1 & 0 & 0 & 0 & 0 \\ 0 & -6 & 1\cdot 11 & -6 & 1 & 0 & 0 & 0 \\ 0 & 24 & -50 & 35 & -10 & 1 & 0 & 0 \\ 0 & -120 & 274 & -225 & 85 & -15 & 1 & 0 \\ 0 & 720 & -1764 & 1624 & -735 & 175 & -21 & 1 \end{pmatrix} = -8 + 11 = 3,$$

and

$$P_{4,5} = (-1)^5 \sum_{k\geq 0}^{4} (-1)^k \binom{5}{k} R_{k,3}$$

$$= -\begin{pmatrix} 1 & 0 & 0 & 1\cdot 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & -5\cdot 0 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 10\cdot 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 2 & -10\cdot 0 & 0 & 0 & 0 & 0 \\ 1 & -6 & 11 & 5\cdot -6 & 0 & 0 & 0 & 0 \\ 1 & -10 & 35 & -1\cdot -50 & 24 & 0 & 0 & 0 \\ 1 & -15 & 85 & -225 & 274 & -120 & 0 & 0 \\ 1 & -21 & 175 & -735 & 1624 & -1764 & 720 & 0 \end{pmatrix} = -(-30 + 50) = -20.$$

Compare this with the polynomials, shifted, reversed A111999, from a symbolic math app generation of the Taylor series of the Borel-Laplace integral rep, from the reduction of A133932, as done above, from the numerouus other Lagrange inversion type formulas in the OEIS (cf. A145271) for compositional inversion applied to $f(x)$, or from the iterated infinigen in matrix or differnetial form:

$$P_1(u) = 1$$

$$P_2(u) = -u^2$$

$$P_3(u) = 2u^3 + 3u^4$$

$$P_4(u) = -6u^4 - 20u^5 - 15u^6$$

$$P_5(u) = 24u^5 + 130u^6 + 210u^7 + 105u^8,$$

or with the coefficients in matrix form

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -6 & -20 & -15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 24 & 130 & 210 & 105 \end{pmatrix}.$$

Note that the shifted, unsigned columns of this matrix are the reversed rows of A008603, or, equivalently, the unsigned diagonals of this matrix are the columns of A08603.

Furthermore, the Borel-Laplace transform can be evaluated in terms of the incomplete gamma function to give an o.g.f. for the shifted row polynomials of A111999.

$$\int_0^\infty \frac{1}{z} e^{-\frac{1}{z}\sigma} [h^{-1}(\sigma;u)]' \, d\sigma = \int_0^\infty \frac{1}{z} e^{-\frac{1}{z}h(\omega;u)} d\omega$$

$$= \int_0^\infty e^{-(1+u)\omega} (1+u\omega z)^{\frac{1}{z}} d\omega = \int_0^\infty e^{-\frac{1+u}{uz}\varpi} (1+\varpi)^{\frac{1}{z}} d\varpi/(uz)$$

$$= \int_1^\infty e^{-\frac{1+u}{u}\frac{1}{z}(\alpha-1)} \alpha^{\frac{1}{z}} d\alpha/(uz) = e^\beta \int_1^\infty e^{-\beta\alpha} \alpha^{\frac{1}{z}} d\alpha/(uz)$$

$$= \frac{1}{uz} \beta^{-(1+\frac{1}{z})} \, e^\beta \Gamma(1+\frac{1}{z},\beta).$$

$$= 1 - u^2 z + (2u^3 + 3u^4) z^2 - ... = P_1(u) + P_2(u) z + P_3(u) z^3 + ...$$

with $\beta = \frac{1+u}{u}\frac{1}{z}$ and the upper incomplete gamma function defined as the Mellin/Laplace integral

$$\Gamma(s,x) = \int_x^\infty e^{-t} t^{s-1} dt = x^s \int_1^\infty e^{-x\alpha} \alpha^{s-1} d\alpha = x^s \int_0^1 e^{-\frac{x}{\nu}} \nu^{-s-1} d\nu \,.$$

Note the similarity to the binomial transforms closely related to Laurent series in the Appendix

$$\frac{1}{uz} \beta^{-(1+\frac{1}{z})} \, e^\beta \Gamma(1+\frac{1}{z},\beta). = \frac{1}{u} \left[ \left( \frac{1+u}{u} \right) \frac{1}{z} \right]^{-\left(\frac{1+z}{z}\right)} \frac{1}{z} e^{\left(\frac{1+u}{u}\right)\frac{1}{z}} \Gamma \left[ \frac{1+z}{z}, \left( \frac{1+u}{u} \right) \frac{1}{z} \right].$$

The general partition polynomials in Part III for the infinigen (in square brackets) for $n > 0$ reduce to the shifted, signed Pascal matrix polynomials $(-1)^n u^{n+1} (1+u)^{n-1}$ for $b_1 = 1$ and $b_n = (-u)^n$ for $n > 1$, i.e., for

$$f(u) = \omega + u\omega - ln(1+u\omega) = h(\omega;u)\,,$$

giving for this special case the o.g.f. for the infinigen

$$g(x) = \frac{1}{b_1 + b_2 x + b_3 x^2 + ...}$$

$$= \frac{1}{1 + u^2 x - u^3 x^2 + u^4 x^3 - ...} = \frac{1}{1 + u - \frac{u}{1+ux}} = \frac{1+ux}{1+(1+u)ux}$$

$$= (1+ux) \sum_{n \geq 0} (-1)^n (1+u)^n (ux)^n$$

$$= \left[ 1 + \sum_{n \geq 1} (-1)^n \left[ (1+u)^n \, u^n - (1+u)^{n-1} \, u^n \right] x^n \right] = 1 + \sum_{n \geq 1} (-1)^n u^{n+1} \, (1+u)^{n-1} x^n$$

$$= D_0[1] + D_1[1, -u] \, x + D_2[1, -u, (-u)^2] \, x^2 + ...$$

with, suppressing arguments, $D_0 = 1$ and, for $n > 0$,

$$D_n = (-1)^n u^{n+1} \, (1+u)^{n-1} = (-1)^n u \sum_{k=0}^{n-1} (-1)^k u^{n-k} D_k \, .$$

The coefficients of the row polynomials for $g(x)$ written in matrix form as shifted, signed rows of the Pascal matrix have shifted columns that are rows of A011973 and A102426, dearated rows of A049310, and shifted rows of A030528; and shifted, alternating columns are the rows of A053122. The partial matrix is (most zeros suppressed)

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & & -1 & & & & & & & & & & \\
0 & & & 1 & 1 & & & & & & & & \\
0 & & & & -1 & -2 & -1 & & & & & & \\
0 & & & & & 1 & 3 & 3 & 1 & & & & \\
0 & & & & & & -1 & -4 & -6 & -4 & -1 & & \\
0 & & & & & & & 1 & 5 & 10 & 10 & 5 & 1 \\
0 & & & & & & & & -1 & -6 & -15 & -20 & -15 \\
0 & & & & & & & & & 1 & 7 & 21 & 35 \\
0 & & & & & & & & & & -1 & -8 & -28 \\
0 & & & & & & & & & & & 1 & 9 \\
0 & & & & & & & & & & & & -1 \\
0 & & & & & & & & & & & &
\end{pmatrix}.$$

# Part VI

### Generalization through umbralization and Appell sequences

Let's add some extra parameters to umbralize the foregoing algorithms and look at

$$f(\omega; x, c, u) = \omega + x\,u\omega - ln(1 + c\,u\omega)$$

$$= [1 + (x - c)u]\omega + \frac{(-cu\omega)^2}{2} + \frac{(-cu\omega)^3}{3} + ...$$

$$= b_1\omega + b_2\frac{\omega^2}{2} + b_3\frac{\omega^3}{3} + ...$$

$$= -ln(1 - b.\omega)\,,$$

which can be inverted using A133932 with the appropriate substitutions for $b_n$. Now umbralize the parameter $c$,

$$f(\omega; x, c., u) = \omega + x\,u\omega - ln(1 + c.\,u\omega)$$

$$= [1 + (x - c.)u]\omega + \frac{(-c.u\omega)^2}{2} + \frac{(-c.u\omega)^3}{3} + ...$$

$$= [1 + (x - c_1)u]\omega + c_2\frac{(-u\omega)^2}{2} + c_3\frac{(-u\omega)^3}{3} + ...$$

$$= b_1\omega + b_2\frac{\omega^2}{2} + b_3\frac{\omega^3}{3} + ...$$

$$= -ln(1 - b.\omega)\,,$$

which again can be inverted using A133932 with the appropriate substitutions for $b_n$:

$$b_1 = 1 + (x - c_1)u \quad \text{and} \quad b_n = c_n(-u)^n \text{ otherwise.}$$

Now look at the compositional inversion formally using the Borel-Laplace transform:

$$\int_0^\infty \frac{1}{z}e^{-\frac{1}{z}\sigma}[f^{-1}(\sigma; x, c., u)]'\,d\sigma = \int_0^\infty \frac{1}{z}e^{-\frac{1}{z}f(\omega;x,c.,u)}d\omega$$

$$= \int_0^\infty \frac{1}{z}e^{-\frac{1}{z}\left\{[1+(x-c_1)u]\omega + c_2\frac{(-u\omega)^2}{2} + c_3\frac{(-u\omega)^3}{3} + ...\right\}}d\omega$$

$$= \int_0^\infty \frac{1}{z}e^{-\frac{1}{z}[\omega + xu\omega - ln(1+c.u\omega)]}d\omega$$

$$= \int_0^\infty e^{-[\omega + xu\omega - \frac{1}{z}ln(1+c.zu\omega)]}d\omega$$

$$= \int_0^\infty e^{-\omega}e^{-xu\omega}exp\left[\frac{1}{z}ln\left(1 + c.zu\omega\right)\right]d\omega.$$

The product in the integrand has the form of an exponential generating function for an Appell Sheffer sequence

$$e^{xt}K(t) = e^{xt}e^{m.t} = e^{(x+m.)t} = e^{A.(x)t}$$

with $K(0) = 1$ and the Appell sequence polynomials

$$A_n(x) = (x + m.)^n = \sum_{k=0}^{n} \binom{n}{k} m_{n-k}x^k = (x + A.(0))^n,$$

implying

$$A_n(x + y) = (A.(x) + y)^n.$$

With $t = u\omega$ the integrand product

$$e^{A(-x;c.,z)t} = e^{-xt}exp\left[\frac{1}{z}ln\left(1 + c.zt\right)\right]$$

$$= e^{-xt}exp\left[-c_1(-t) - c_2z\frac{(-t)^2}{2} - c_3z^2\frac{(-t)^3}{3} - ...\right]$$

$$= exp\left[\left(c_1 - x\right)t - c_2z\frac{t^2}{2} + c_3z^2\frac{t^3}{3} + ...\right]$$

$$= exp\left[-ln(1 - \hat{c}.t)\right] = exp\left[\hat{c}_1t + \hat{c}_2\frac{t^2}{2} + \hat{c}_3\frac{t^3}{3} + ...\right]$$

has the form of the the exponential generating function for the cycle index polynomials for the symmetric groups, the partition polynomials of the refined Stirling numbers of the first kind $St1_n[\hat{c}_1, \hat{c}_2, ..., \hat{c}_n]$ of A036039 (sequence invertible by the Faber polynomials A263916), with

$$\hat{c}_1 = c_1 - x \quad \text{and} \quad \hat{c}_n = (-1)^{n-1}z^{n-1}c_n \text{ otherwise,}$$

which is an Appell sequence in the indeterminate $\hat{c}_1$, and it follows that

$$A_n(-x; c_1, c_2, ..., c_n, z) = A_n(0; c_1 - x, c_2, ..., c_n, z) = [A.(0; c_1, c_2, ..., c_n, z) - x]^n,$$

so we need look at the moment sequence only to characterize the form of the partition polynomials:

$$exp\left[\frac{1}{z}ln\left(1 + c.zt\right)\right] = e^{tA.(0;c_1,c_2,...,c_n,z)}$$

$$= 1 + [c_1]\,t + \left[-1zc_2 + 1c_1^2\right]\frac{t^2}{2!} + \left[2z^2c_3 - 3zc_1c_2 + 1c_1^3\right]\frac{t^3}{3!}$$

$$+ \left[-6z^3c_4 + 3z^2c_2^2 + 8z^2c_1c_3 - 6zc_1^2c_2 + 1c_1^4\right]\frac{t^4}{4!} + ...$$

(for a compilation of $St1_n$, see the Lang link in the OEIS entry). The polynomials graded in $c_1$with the other parameters unity give the row polynomials of the inverse A055137 of the recontres number triangle A008290, which begins as

$$\begin{pmatrix} 1 & & & & \\ 0 & 1 & & & \\ -1 & 0 & 1 & & \\ 2 & -3 & 0 & 1 & \\ -3 & 8 & -6 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ c_1 \\ c_1^2 \\ c_1^3 \\ c_1^4 \end{pmatrix},$$

and, when graded in $z$ with the other parameters unity, give the reverse row polynomials of the Stirling polynomials of the first kind as above. The first full partition polynomials for $A.(0; c_1, c_2, ..., c_n, z)$ graded in $z$ are in matrix form

$$\begin{pmatrix} 1 & & & & & \\ c_1 & 0 & & & & \\ c_1^2 & -c_2 & 0 & & & \\ c_1^3 & -3c_1c_2 & 2c_3 & 0 & & \\ c_1^4 & -6c_1^2c_2 & (8c_1c_3 + 3c_2^2) & -6c_4 & 0 & \\ c_1^5 & -10c_1^3c_2 & (15c_1c_2^2 + 20c_1^2c_3) & -(20c_2c_3 + 30c_1c_4) & 24c_5 & \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \\ z^4 \end{pmatrix}.$$

Returning to the Borel-Laplace transform then,

$$\int_0^\infty \frac{1}{z} e^{-\frac{1}{z}\sigma} [f^{-1}(\sigma; x, c., u)]' \, d\sigma = \int_0^\infty \frac{1}{z} e^{-\frac{1}{z} f(\omega; x, c., u)} d\omega$$

$$= \int_0^\infty e^{-\omega} e^{-xu\omega} exp \left[ \frac{1}{z} ln \left(1 + c.zu\omega\right) \right] d\omega$$

$$= \int_0^\infty e^{-\omega} e^{u\omega A.(-x; c_1, c_2, ..., c_n, z)} d\omega$$

$$= \frac{1}{1 - uA.(-x; c_1, c_2, ..., c_n, z)} = \sum_{n \geq 0} A_n(-x; c_1, c_2, ..., c_n, z) \, u^n$$

$$= \sum_{n \geq 0} A_n(0; c_1 - x, c_2, ..., c_n, z) \, u^n$$

$$= \sum_{n \geq 0} [1 - A.(1; c_1 - x, c_2, ..., c_n, z)]^n \; (-u)^n$$

$$= \sum_{n \geq 0} [1 - A.(0; 1 + c_1 - x, c_2, ..., c_n, z)]^n \; (-u)^n,$$

which has the form of the earlier equations with the polynomials of the matrix $R$ of the reversed Stirling polynomials. The corresponding matrix graded in $z$ for the cycle index partition polynomials $A_n(0;\hat{c}_1, c_2, ..., c_n, z)$ with $\hat{c}_1 = 1 + c_1 - x$ is partially represented again for ease of reference for the following formulas

$$\begin{pmatrix} 1 & & & & \\ \hat{c}_1 & 0 & & & \\ \hat{c}_1^2 & -c_2 & 0 & & \\ \hat{c}_1^3 & -3\hat{c}_1 c_2 & 2c_3 & 0 & \\ \hat{c}_1^4 & -6\hat{c}_1^2 c_2 & (8\hat{c}_1 c_3 + 3c_2^2) & -6c_4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \\ z^4 \end{pmatrix},$$

and the same arguments for convolutions along the columns apply as for $R$ to give the elements of the compositional inversion of A133932; e.g.,

$$P_{3,3} = (-u)^3 \begin{pmatrix} 1 & 0 & 1 \cdot 0 & 0 & 0 \\ \hat{c}_1 & 0 & -3 \cdot 0 & 0 & 0 \\ \hat{c}_1^2 & -c_2 & 3 \cdot 0 & 0 & 0 \\ \hat{c}_1^3 & -3\hat{c}_1 c_2 & -1 \cdot 2c_3 & 0 & 0 \\ \hat{c}_1^4 & -6\hat{c}_1^2 c_2 & 1 \cdot (8\hat{c}_1 c_3 + 3c_2^2) & -6c_4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \\ z^4 \end{pmatrix}$$

$$= (-u)^3 \left[-2c_3\right] z^2 = 2u^3\, c_3\, z^2$$

and

$$P_{3,4} = (-u)^4 \begin{pmatrix} 1 & 0 & 1 \cdot 0 & 0 & 0 \\ \hat{c}_1 & 0 & -4 \cdot 0 & 0 & 0 \\ \hat{c}_1^2 & -c_2 & 6 \cdot 0 & 0 & 0 \\ \hat{c}_1^3 & -3\hat{c}_1 c_2 & -4 \cdot 2c_3 & 0 & 0 \\ \hat{c}_1^4 & -6\hat{c}_1^2 c_2 & 1 \cdot (8\hat{c}_1 c_3 + 3c_2^2) & -6c_4 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \\ z^4 \end{pmatrix}$$

$$= u^4 \left[-8c_3 + (8\hat{c}_1 c_3 + 3c_2^2)\right] z^2$$

$$= u^4 \left[-8c_3 + 8(1 + (c_1 - x))\, c_3 + 3c_2^2\right] z^2.$$

Compare this with the polynomial for the third order $z^3$ from A133932

$$\widetilde{P}_3(b_1, b_2, b_3) = \frac{1}{b_1^5} \left(3b_2^2 - 2b_1 b_3\right)$$

$$= P_3(x; c_1, c_2, c_3) = \frac{1}{\left[1 + (x - c_1)u\right]^5} \left\{3c_2^2 u^4 + 2\left[1 + (x - c_1)u\right] c_3 u^3\right\}$$

with

$$b_1 = 1 + (x - c_1)u \quad \text{and} \quad b_n = c_n (-u)^n \text{ otherwise.}$$

For $c_1 = x$, then $b_1 = 1 + (x - c_1)u = 1$, and $\hat{c}_1 = 1 + c_1 - x = 1$, and, in agreement with earlier sections, the third order term of the compositional inverse is

$$P_3(x; c_1 = x, c_2, c_3)\, \frac{z^3}{3!} = P_3(0; 0, c_2, c_3)\, \frac{z^3}{3!} = \left[3c_2^2 u^4 + 2c_3 u^3\right] \frac{z^3}{3!}$$

$$= \left[P_{3,3} + P_{3,4}\right] z\, \frac{1}{3!} = \left\{\left[2u^3 c_3 z^2\right] + \left[u^4 \left[-8c_3 + 8c_3 + 3c_2^2\right] z^2\right]\right\} z\, \frac{1}{3!}.$$

For the general case with $b_1 = 1 + (x - c_1)u$ and $\hat{c}_1 = 1 + c_1 - x$, let's check that the direct inversion and the binomial transforms of the first column give the same result formally. From the direct inversion, the first order term of the inversion is

$$P_1(b_1)\, z = \frac{1}{b_1} z,$$

and the contributions from the binomial transforms of the first column of the cycle index polynomial array are

$$P_{1,0} = (-u)^0 1$$

$$P_{1,1} = (-u)^1 (1 - \hat{c}_1)$$

$$P_{1,2} = (-u)^2 (1 - 2\hat{c}_1 + \hat{c}_1^2) = (-u(1 - \hat{c}_1))^2$$

and so on with

$$P_{1,n} = (-u(1 - \hat{c}_1))^n.$$

Summing the contributions gives consistently

$$\sum_{n \geq 0} P_{1,n} = \sum_{n \geq 0} (-u(1-\hat{c}_1))^n = \frac{1}{1 + u(1 - \hat{c}_1)} = \frac{1}{1 + u(1 - (1 + c_1 - x))} = \frac{1}{1 + u(x - c_1)} = \frac{1}{b_1}.$$

Note that the o.g.f. for the first column is

$$\frac{1}{1 - \hat{c}_1 y} = 1 + \hat{c}_1 y + (\hat{c}_1 y)^2 + \cdots$$

and that summing over the binomial transforms is equivalent to replacing $\hat{c}_1 y$ by $-u(1 - \hat{c}_1)$.

Similarly for the second order term,

$$P_2(b_1, b_2) = \frac{1}{b_1^3}\, [-b_2],$$

and the second column of the cycle index partition array has the o.g.f.

$$-c_2 \frac{y^2}{(1 - \hat{c}_1 y)^3} = -c_2 y^2 (1 + 3\hat{c}_1 y + 6\hat{c}_1^2 y^2 + 10\hat{c}_1^3 y^3 + \cdots).$$

A few binomial transform calculations on the this column show that the transforms' actions are equivalent to replacing $\hat{c}_1 y$ by $-u(1 - \hat{c}_1)$ and the $y^2$ factor by $(-u)^2$ in the o.g.f., giving, as it must if the general argument is correct,

$$-c_2 \frac{(-u)^2}{(1 + u(1 - \hat{c}_1))^3} = -\frac{b_2}{b_1^3} = P_2(b_1, b_2).$$

An example of the contribution of a single finite difference on the second column is

$$P_{2,5} = (-u)^5(1{\cdot}0{+}{-}5{\cdot}0{+}10{\cdot}1{-}10{\cdot}3\hat{c}_1{+}5{\cdot}6\hat{c}_1^2{-}1{\cdot}10\hat{c}_1^3) = (-u)^2 10\left[-u(1-\hat{c}_1)\right]^3 .$$

The other columns are not so easy to characterize, so we must fall back on the general argument; however, these observations as well as spot checks on the vanishing of binomal transforms on the other columns when $x - c_1 = 0$ , i.e., $b_1 = \hat{c}_1{=}1,$ lend credence to the general argument.

Relating the two different methods for obtaining the compositional inverse provides a way of identifying convolutions, represented by multiplication of $1/[1 + (x - c_1)u]^{2n+1}$ and polynomials, with finite differences of columns or diagonals of number triangles, as exemplified above. Application to inversion of $f(z) = z{+}\frac{t}{n}\,z^n$ provides connections to the Fuss-Catalan numbers (see "Discriminating deltas, depressed equations, and generalized Catalan numbers").

Compare these results with Theorem 1.10.7 on page 64 of the dissertation "An inversion theorem for labeled trees and some limits of areas under lattice paths" by Brian Drakes (see also Appendix II below and Peter Bala's link in A111999). As an illustration of the connections, consider Example 1.10.8 on the Narayana polynomials of Drake's thesis. Identify

$$f(\omega) = \omega - \frac{t\omega}{1-\omega} = (1-t)\omega - t\omega^2 - t\omega^3 - t\omega^4 - ...$$

$$= (1-t)\omega - 2t\frac{\omega^2}{2} - 3t\frac{\omega^3}{3} - 4t\frac{\omega^4}{4} - ...$$

$$= b_1\omega + b_2\frac{\omega^2}{2} + b_3\frac{\omega^3}{3} + ... = -ln(1 - b.\omega)$$

with

$$b_1 = (1-t) = 1 + (x - c_1)u \quad \text{and} \quad b_n = -n\,t = c_n(-u)^n \text{ otherwise.}$$

Then letting $u = 1$, $(x - c_1) = -t$ and $\hat{c}_1 = 1 + c_1 - x = (1+t)$, $c_n = (-1)^{n+1}nt$ for $n > 0$,

$$\widetilde{P}_3(b_1, b_2, b_3)\frac{z^3}{3!} = \frac{1}{b_1^5}\left(3b_2^2 - 2b_1b_3\right)\frac{z^3}{3!} = \frac{1}{(1-t)^5}\left\{3(-2t)^2 - 2\,(1-t)\,(-3t)\right\}\frac{z^3}{3!}$$

$$= P_3(x; c_1, c_2, c_3)\frac{z^3}{3!} = \frac{1}{\left[1 + (x - c_1)u\right]^5}\left\{3c_2^2u^4 + 2\left[1 + (x - c_1)u\right]c_3u^3\right\}\frac{z^3}{3!}$$

$$= \frac{1}{[1-t]^5} \left\{3\,(-2t)^2 + 2\,[1-t]\,(3t)\right\} \frac{z^3}{3!}$$

$$= \frac{1}{[1-t]^5}\,(t+t^2)\,z^3 = (t + 6t^2 + 20t^3 + 50t^4 + 105t^5 + ...)\,z^3\,,$$

so the numerator of the rational fraction, $(t+t^2)$, is the third row of the Narayana triangle of A090181 (A001263), and the final product is the shifted fourth column and the third diagonal.

As another illustration of the algorithm, consider shifted, reversed A028246 with e.g.f. (omitting the first row)

$$\omega = G_E(z;t) = e^{p.(t)z} = \frac{1}{t} - \frac{1}{t + (1-t)(1-e^{-t^2 z})}$$

$$= (1-t)\,z + (-2t + 3t^2 - t^3)\,\frac{z^2}{2!} + (6t^2 - 12t^3 + 7t^4 - t^5)\frac{z^4}{3!} + \cdots$$

and compositional inverse in $z$ about the origin

$$z = G_E^{-1}(\omega, t) = -\frac{1}{t^2} ln\left[\frac{1-t(1+\omega)}{(1-t)(1-t\omega)}\right] = f(\omega, t)$$

$$= \frac{1}{1-t}\,\omega + \frac{2t-t^2}{(1-t)^2}\frac{\omega^2}{2} + \frac{3t^2-3t^3+t^4}{(1-t)^3}\frac{\omega^3}{3} + \frac{4t^3-6t^4+4t^5-t^6}{(1-t)^4}\frac{\omega^4}{4} + \cdots$$

with the polynomial numerators $P_n(t)$ in the rational function being the shifted, signed rows of A135278 (reversed A074909), so let

$$b_1 = 1 + (x - c_1)u = \frac{1}{1-t} \quad \text{and} \quad u = \frac{1}{1-t}\,,$$

implying

$$x - c_1 = \frac{1}{u}\frac{t}{1-t} = t$$

and

$$\hat{c}_1 = 1 + c_1 - x = 1{-}t\,.$$

Then identify for $n \geq 0$

$$b_n = \frac{P_n(t)}{(1-t)^n} = (-1)^n c_n u^n\,,$$

i.e.,

$$c_n = (-1)^n P_n(t)\,.$$

The rational functions $b_n$ from the general arguments above are the columns of A074909:

$$\frac{P_1(t)}{(1-t)} = \frac{1}{1-t} = 1 + t + t^2 + t^4 + t^5 + \cdots$$

$$\frac{P_2(t)}{(1-t)^2} = \frac{2t-t^2}{(1-t)^2} = 2t + 3t^2 + 4t^4 + 5t^5 + \cdots$$
$$\frac{P_3(t)}{(1-t)^3} = \frac{3t^2-3t^3+t^4}{(1-t)^3} = 3t^2 + 6t^3 + 10t^4 + 15t^5 + \cdots$$
$$\frac{P_4(t)}{(1-t)^4} = \frac{4t^3-6t^4+4t^5-t^6}{(1-t)^4} = 4t^3 + 10t^4 + 20t^5 + 35t^6 + \cdots .$$

and A074909 begins

$$\begin{pmatrix} 1 & & & & \\ 1 & 2 & & & \\ 1 & 3 & 3 & & \\ 1 & 4 & 6 & 4 & \\ 1 & 5 & 10 & 10 & 5 \end{pmatrix}.$$

The compositional inversion directly using A133932 gives for the first few partition polynomials for the inverse of $f(\omega)$ the row polynomials of $G_E(z,t)$ :

$$Prt_1(b_1) = \frac{1}{b_1}[1] = (1-t)$$

$$Prt_2(b_1, b_2) = \frac{1}{b_1^3}[-b_2] = (1-t)(2t-t^2) = 2t - 3t^2 + t^3$$

$$Prt_3(b_1, b_2, b_3) = \frac{1}{b_1^5}[3b_2^2 - 2b_1b_3] = (1-t)(6t^2 - 6t^3 + t^4)$$
$$= 6t^2 - 12t^3 + 7t^4 - t^5$$

$$Prt_4(b_1, b_2, b_3, b_4) = \frac{1}{b_1^7}[-15b_2^3 + 20b_1b_2b_3 - 6b_1^2b_4]$$
$$= (1-t)(-24t^3 + 36t^4 - 14t^5 + t^6)$$
$$= -24t^3 + 60t^4 - 50t^5 + 15t^6 - t^7$$

with the intermediate appearance of A090582 the signed, shifted, reverse face polynomials of the permutahedra A019538 All these polynomial sequences, together with the Eulerian polynomials A008292, are relatable through simple transformations, reflected in the simple relation between the mutually inverse $e^x - 1$ and $ln(1+x)$. The compostional inverses for these polynomial sequences can be couched as the logarithm of a rational function (i.e., a ratio of polynomials), giving rise to a quadratic infinigen.

For this example. the first full partition polynomials for $A.(0; \hat{c}_1 = 1, c_2, ..., c_n, z)$ graded in $z$ are

$$\begin{pmatrix} 1 & & & & & \\ 1-t & 0 & & & & \\ (1-t)^2 & -c_2 & 0 & & & \\ (1-t)^3 & -3c_2(1-t) & 2c_3 & 0 & & \\ (1-t)^4 & -6c_2(1-t)^2 & (8c_3 + 3(1-t)c_2^2) & -6c_4 & 0 & \\ (1-t)^5 & -10c_2(1-t)^3 & (15c_2^2 + 20(1-t)c_3) & -(20c_2c_3 + 30(1-t)c_4) & 24c_5 & \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \\ z^4 \end{pmatrix}.$$

The binomial transforms of the first and second columns may be summed as above for the general case.

## Appendix I: Some additional material on the Euler-binomial transforms

The umbral calculus and the Euler-binomial trnasforms fit each other hand-in-glove. The transforms are compactly ensconsed in a set of differential operators representing generalized shift operators that are equivalent when acting on certain sets of functions analytic at the origin:

$$e^{a.:xD_{x=0}:} = e^{-(1-a):xD_x:} = e^{-b.:xD_x:}$$

$$= \sum_{n\geq 0} \frac{(a. : xD_{x=0} :)}{n!} = \sum_{n\geq 0} a_n \frac{x^n D_{x=0}^n}{n!}$$

$$= \sum_{n\geq 0} \frac{(-(1-a.) : xD_x :)^n}{n!} = \sum_{n\geq 0} (-1)^n (1-a.)^n \frac{x^n D_x^n}{n!} = \sum_{n\geq 0} (-1)^n b_n \frac{x^n D_x^n}{n!},$$

where

$$(1-a.)^n = \sum_{k\geq 0}^{n} (-1)^k \binom{n}{k} a_k = \nabla_{k=0}^n a_k = b_n$$

is the involutive binomial transform (or finite differences) of the sequence $a_n$.

Actions on $e^x$ and on $1/(1-x)$ generate the Euler transforms for e.g.f.s,

$$e^{a.:xD_{x=0}:} e^x = e^{-(1-a):xD_x:} e^x = e^{-b.:xD_x:} e^x$$

$$= e^{a.x} = e^{x-(1-a.)x} = e^x e^{-(1-a.)x} = e^{(1-b.).x} = e^x e^{-b.x},$$

and for o.g.f.s,

$$e^{a.:xD_{x=0}:} \frac{1}{1-x} = e^{-(1-a):xD_x:} \frac{1}{1-x} = e^{-b.:xD_x:} \frac{1}{1-x}$$

$$= \frac{1}{1-a.x} = \frac{1}{1-[x-(1-a)x]} = \frac{1}{1-(x-b.x)}$$

$$= \frac{1}{1-x} \frac{1}{1+(1-a.)\left(\frac{x}{1-x}\right)} = \frac{1}{1-(1-b.)x} = \sum_{n\geq 0} (1-b.)^n x^n$$

$$= \sum_{n\geq 0} a_n x^n = -\frac{1}{x} \sum_{n\geq 0} (1-a.)^n \left(\frac{x}{x-1}\right)^{n+1} = -\frac{1}{x} \sum_{n\geq 0} b_n \left(\frac{x}{x-1}\right)^{n+1}.$$

(Check with $a_n = (1+t)^n$.) Note that the involutive property of the Euler-binomial transform for the o.g.f. is reflected in the self-inverse character of $x/(x-1)$ and that, for the e.g.f.s, both the e.g.f. and its Euler-binomial transform are convergent if one is, whereas this is typically not true of the o.g.f. and its transform (act on $x^\alpha$, giving generalized finite differences involving the binomial coefficients $\binom{\alpha}{n}$).

Changing variables in the last equation gives relations involving the row polynomials $Psc_n(t) = (1+t)^n$ of the Pascal matrix A007318:

$$\sum_{n\geq 0} a_n (1+t)^n = -\frac{1}{t} \sum_{n\geq 0} b_n \frac{(1+t)^n}{t^n} = -\frac{1}{t} \sum_{n\geq 0} b_n \left(\frac{1+t}{t}\right)^n$$

$$= \sum_{n\geq 0} a_n Psc_n(t) = -\frac{1}{t} \sum_{n\geq 0} b_n \frac{Psc_n(t)}{t^n}$$

$$= \frac{1}{1-a.Psc.(t)} = \frac{1}{1-a.(1+t)} = \frac{1}{b.(1+t)-t} = -\frac{1}{t} \frac{1}{1-b.Psc.(t)/t}.$$

For $a_n = (1-x)^n$, these become

$$\sum_{n\geq 0} (1-x)^n (1+t)^n = -\frac{1}{t} \sum_{n\geq 0} x^n \left(\frac{1+t}{t}\right)^n$$

$$= \frac{1}{1-(1-x)(1+t)} = -\frac{1}{t} \frac{1}{1-x\left(\frac{1+t}{t}\right)} = \frac{1}{-t+x(1+t)}$$

$$= \sum_{n\geq 0} Psc_n(-x)\, Psc_n(t) = -\frac{1}{t} \sum_{n\geq 0} \left(\frac{x\, Psc.(t)}{t}\right)^n = -\frac{1}{t} \sum_{n\geq 0} \left(\frac{x}{t}\right)^n Psc_n(t)$$

$$= \frac{1}{1-Psc.(-x)\, Psc.(t)} = -\frac{1}{t} \frac{1}{1-x\, Psc.(t)/t}.$$

The first finite differences of the coefficients of this o.g.f. in $x$ are often enlightening. For a sequence $a_n$, these differences are given by convolving the o.g.f. with $(1-x)$:

$$\frac{1-x}{1-a.x} = (1-x) \sum_{n\geq 0} a_n x^n = a_0 + \sum_{n\geq 1} (a_n - a_{n-1})\, x^n = \sum_{n\geq 0} d_n\, x^n.$$

The coefficients can be regenerated by taking the partial sums of the finite differences ( i.e., deconvolving the difference sequence by multiplying by $1/(1-x)$) since

$$s_n = d_o + d_1 + ... + d_n = a_0 + a_1 - a_0 + a_2 - a_1 + ... + a_n - a_{n-1} = a_n.$$

So, applying these maneuvers to an example o.g..f. (compare with the compositional inverse of a shifted o.g.f. of the signed Narayana polynomials A001263),

$$\frac{1}{-(1+t)+\frac{1}{1-x}} = (1-x)\,\frac{1}{1-(1-x)(1+t)} =$$

$$= \sum_{n\geq 0}(1-x)^{n+1}(1+t)^n = (1-x)\sum_{n\geq 0} -\frac{1}{t}\left(\frac{1+t}{t}\right)^n x^n$$

$$= -\frac{1}{t}\left[1+\sum_{n\geq 1}\left[\left(\frac{1+t}{t}\right)^n - \left(\frac{1+t}{t}\right)^{n-1}\right]x^n\right] = -\frac{1}{t}+\sum_{n\geq 1}-\frac{1}{t^2}\left(\frac{1+t}{t}\right)^{n-1} x^n.$$

Then the first finite differences of $c_n = -\frac{1}{t}\left(\frac{1+t}{t}\right)^n$ are $-\frac{1}{t}, -\frac{1}{t^2}, -\frac{1}{t^2}\frac{1+t}{t}, \ldots, -\frac{1}{t^2}\left(\frac{1+t}{t}\right)^{n-1}, \ldots$, i.e., $d_0 = -\frac{1}{t}$ and $d_n = -\frac{1}{t^2}\left(\frac{1+t}{t}\right)^{n-1} = \frac{1}{t}c_{n-1}$ for $n > 0$, and the partial sums of the differences give

$$c_n = -\frac{1}{t}\left(\frac{1+t}{t}\right)^n = d_0 + d_1 + \ldots + d_n$$

$$= -\frac{1}{t} - \frac{1}{t^2} - \frac{1}{t^2}\left(\frac{1+t}{t}\right)^1 - \frac{1}{t^2}\left(\frac{1+t}{t}\right)^2 \ldots - \frac{1}{t^2}\left(\frac{1+t}{t}\right)^{n-1}$$

$$= \frac{1}{t}(-1 + c_0 + c_1 + \ldots + c_{n-1}),$$

which can be confirmed by using the sum of a geometric series.

Returning to the (slightly rewritten) Euler-binomial transform for an o.g.f.

$$\sum_{n\geq 0} a_n x^n = \sum_{n\geq 0}(a.-1)\,\frac{x^n}{(1-x)^{n+1}} = \sum_{n\geq 0}(-1)^n\, b_n\,\frac{x^n}{(1-x)^{n+1}},$$

notice that $\frac{x^n}{(1-x)^{n+1}}$ is the o.g.f. for the n-th column of the Pascal matrix and that $\frac{1}{(1-x)^{n+1}}$ is that for the n-th diagonal (holds for all palindromic lower triangular matrices).

There are myriad simple associations among the shift, binomial, and Laplace transforms and the Laguerre and Euler differential ops. Some are

$$\frac{1}{(1-u)}\,\frac{1}{1+a.\frac{u}{1-u}} = \frac{1}{1-(1-a.)u}$$

$$= \sum_{n\geq 0}(-1)^n a_n\,\frac{u^n}{(1-u)^{n+1}} = \sum_{n\geq 0}(1-a)^n u^n$$

$$= \int_0^\infty \frac{1}{1-u} e^{-\omega(1+a.\frac{u}{1-u})} d\omega = \int_0^\infty e^{-\omega} \frac{1}{1-u} e^{-a.\omega \frac{u}{1-u}} d\omega \,.$$

$$= \int_0^\infty e^{-\omega} \sum_{n\geq 0} L_n(a.\omega)\, u^n d\omega = \sum_{n\geq 0} \left[ \int_0^\infty e^{-\omega} L_n(a.\omega)\, d\omega \right] u^n$$

$$= \sum_{n\geq 0} u^n L_n(a.D_{u=0}) \int_0^\infty e^{-\omega} e^{\omega u} d\omega = \sum_{n\geq 0} u^n L_n(a.D_{u=0}) \frac{1}{1-u}$$

$$= \sum_{n\geq 0} (1-a.)^n \frac{u^n D_{u=0}^n}{n!} \frac{1}{1-u} = \sum_{n\geq 0} (-1)^n a_n \frac{u^n D_u^n}{n!} \frac{1}{1-u}$$

$$= e^{(1-a.):uD_{u=0}:} \frac{1}{1-u} = e^{-a.:uD_u:} \frac{1}{1-u}$$

$$= \sum_{n\geq 0} (1-a.)^n (1-L.(:uD_{u=0}:)^n \frac{1}{1-u} = \sum_{n\geq 0} (-a.)^n (1-L.(:uD_u:)^n \frac{1}{1-u}$$

$$= \sum_{n\geq 0} (-1)^n a_n \frac{u^n D_u^n}{n!} \frac{1}{1-u} = \sum_{n\geq 0} (-1)^n a_n \binom{uD_u}{n} \frac{1}{1-u} = (1-a.)^{uD_u} \frac{1}{1-u}$$

$$= exp\left[uD_u ln(1-a.)\right] \frac{1}{1-u}$$

where

$$L_n(x) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{x^k}{k!} = \nabla_{k=0}^n \frac{x^k}{k!}$$

is a Laguerre polynomial and, since the finite differences are involutive, i.e., $\nabla_{k=0}^n \nabla_{j=0}^k c_j = c_n$,

$$(1-L.(x))^n = \frac{x^n}{n!}.$$

(Be careful to evaluste the differential op products of the form $(:AB:)^n(:AB:)^m$ as $(:AB:)^{n+m} = A^{m+n}B^{m+n}$and not as $A^nB^nA^mB^m$; that is, simplify all expression containing such ops to a power series with the reduced monomials $:AB:^k$ before evaluating them umbrally. Evaluation at other levels must be explicitly introduced unless unambiguuously obvious.)

The expression $exp\left[uD_u ln(1-a.t)\right]$ can be related to A036039, a refinement of the Stirling polynomials of the first kind, tying the arguments here to more general relations.

## Appendix II: Comparisons with formulas in Drake's thesis

Consider, as Drake did,

$$f(\omega) = \omega - t\, F(\omega)\,,$$

where,

$$F(x) = e^{a.x} - a_0$$

is an e.g.f. so that $F(0) = 0 = f(0)$ and $F$ and $f$ are invertible about the origin. Then

$$e^{zf(\omega)} = e^{\omega S.(z)} = e^{z(\omega - tF(\omega))} = e^{z\omega} e^{-ztF(\omega)} = e^{\omega z} e^{\omega T.(-zt)} = e^{\omega(z - T.(-zt))},$$

where $S_n(z)$ and $T_n(z)$ are binomial Sheffer polynomials. So,

$$S_n(z) = [z - T.(-zt)]^n = z^n \left[1 - \frac{1}{z} T.(-zt)\right]^n = z^n \nabla_k^n z^{-k} T_k(-zt) \,,$$

or

$$z^n S_n(\frac{1}{z}) = R_n(z) = \nabla_k^n z^k T_k(-\frac{t}{z}) = \nabla_k^n U_k(z,t) = [1 - U.(z,t)]^n \,,$$

where $R_n(z)$ is the reverse of $S_n(z)$; $U_n(z,-1)$, the reverse of $T_n(z)$; and $U_n(z,t)$, the binomial transform of $R_n(z)$.

Applying the Borel-Laplace transform as above,

$$\int_0^\infty \frac{1}{z} e^{-\frac{1}{z}\sigma} [f^{-1}(\sigma)]' d\sigma = \int_0^\infty \frac{1}{z} e^{-\frac{1}{z} f(\omega)} d\omega$$

$$= \int_0^\infty \frac{1}{z} e^{-\frac{1}{z}[\omega - t\,F(\omega)]} d\omega = \int_0^\infty \frac{1}{z} e^{-\frac{\omega}{z}} e^{\omega T.(\frac{t}{z})} d\omega$$

$$= \int_0^\infty e^{-\omega} e^{\omega z T.(\frac{t}{z})} d\omega = \int_0^\infty e^{-\omega[1 - U.(z,t)]} d\omega$$

$$= \frac{1}{1 - U.(z,t)} = \sum_{n \ge 0} U_n(z,t) = \sum_{n \ge 0} (1 - R.(z))^n \,,$$

so the coefficients of the e.g.f. of the formal compositional inverse of $f$ are determined by the binomial transforms of the reversed row polynomials $R.$ of the Sheffer sequence $S.$ associated to $f$. Teasing out the relation of these coefficients to the sum as in Part V reveals that the coefficients are the sum of the binomial transforms of the columns of the matrix rep of the $R.$ polynomials, or equivalent binomial transforms of the diagonals of the matrix rep for the $S.$ polynomials.

The fact that the generating functions of the diagonals, or the columns, of the matrix rep of the reverse polynomials have coefficients that are rational functions, or ratio of polynomials, follows directly from the various, graded Lagrange-inversion-type formulas for the compositional inverse compiled in A145271, of which A133932 is only one rep. They all have at each grade the divisor $1/b_1^{2n+1}$, which for the function being inverted by Drake, $f(x) = x - tF(x)$, is $1/(1 - a_1 t)^{2n+1}$ with $a_1 = F'(0)$.

## Appendix III: Roadmap

The notes Lagrange a la Lah Part I show how umbralization of basic composition formulas and application of the Borel-Laplace transform lead to related yet different Lagrange inversion formulas (LIF). Each formula has at its roots an iconic functional form with an associated binomial Sheffer sequence:

| Iconic Fct. | Sheffer E.g.f. | Asoc. Prt. Poly. | Asc. LIF | Reduced Poly. RP | Reverse RP, RR | *Diag. are RR* | h − vect. for RR |
|---|---|---|---|---|---|---|---|
| $ln(1+x)$ | factorials $e^{t\,ln(1+x)}$ A008275 Revert A008276 | cycle index $e^{-t\,ln(1-c.x)}$ A036039 | A133932 | A111999 | A259456 | A008306 | A112007 Revert A008517. Numerat. for diag. of A008275. |
| $e^x - 1$ | Bell $e^{t\,(e^x-1)}$ A008277 Revert A008278 | gen. Bell $e^{t\,(e^{c.x}-c_0)}$ A036040 | A134685 | A181996 | A134991 | A008299 | A008517 Revert A112007. Numerat. for diag. of A048993 A008277 |
| $\frac{x}{x-1}$ | Lah $e^{t\,\frac{x}{1-x}}$ A008297 Revert A089231 | gen. Lah $e^{t\,\frac{c.x}{1-c.x}}$ A130561 | A133437 | A126216 scaled by n! | A086810 A033282 scaled | A132081 scaled | Scaled A001263 self−revert. Numerat. for diag. of A001263 |

See Lagrange à la Lah Part I for more info. $h(x) = e^x - 1$ and $h^{-1}(x) = ln(1+x)$ are a compositional inverse pair, so the matrices of coefficients for the associated binomial Sheffer sequences $e^{tg(x)} = e^{b.(x)t}$ , are multiplicative inverses. $x/(x-1)$ is self-inverse, so its associated matrix is self-inverse. The Lah polynomials are also the normalized Laguerre polynomials of order -1. These are all core matrices in enumerative/geometric/graph combinatorics, special functions/Lie theory, umbral calculus/differential operators, and mathematical physics.

Links for the core arrays (see also below):

1) Stirling #s first kind: A008275, A008276, A036039, A094638, A130534
   a) Assoc. LIF: A133932
   b) Reduced LIF (R) and its reverse (RR): A111999, A259456
   c) Assoc. formal H polynomials: A112007
   d) Matrix whose diagonals are rows of reversed, reduced LIF (RR): A008306
2) Stirling #s second kind: A008277, A008278, A036040 ,A048993
   a) Assoc. LIF: A134685
   b) Reduced LIF (R) and its reverse (RR): A181996, A134991
   c) Assoc. formal H polynomial: A008517
   d) Matrix whose diagonals are rows of reversed, reduced LIF (RR): A008299
3) Lah #s: A008297, A066667, A089231, A105278, A111596, A130561
   a) Assoc.LIF: A133437 (related to associahedra and dissection of polygons)
   b) Reduced LIF (R) and its reverse: scaled A126216, A086810, A033282)
   c) Assoc formal H polynmial: scaled A001263 (Narayana numbers)
   d) Matrix whose diag. are rows of RR: scaled A132081

The formal h-vectors $H_n(t)$ are related analytically to the reversed, reduced LIF polynomials $RR_n(t)$ by

$$\frac{H_n(t)}{(1-t)^{n+1}} = \frac{1}{1-t} RR_n\left(\frac{t}{1-t}\right)$$

and to the diagonals of the associated Sheffer polynomial matrix $S$ and, therefore, the shifted columns of $R$, the reverse of $S$, by

$$\frac{H_n(t)}{(1-t)^{2n+1}} = n-th\, diagonal\, of\, S\ = n-th\, shifted\, column\, of\, R\,.$$

with the indexing of the matrix reps below initialized at $(n=0, k=0)$

### A) Stirling polynomials of the first kind (falling factorials)

Stirling numbers of the first kind, $S$ matrix, padded A008275, A132393:

$$\begin{pmatrix} 1 & & & & & & \\ 0 & 1 & & & & & \\ 0 & 1 & 1 & & & & \\ 0 & 2 & 3 & 1 & & & \\ 0 & 6 & 11 & 6 & 1 & & \\ 0 & 24 & 50 & 35 & 10 & 1 & \\ 0 & 120 & 274 & 225 & 85 & 15 & 1 \end{pmatrix}$$

Revert $R$, A008276, of $S$:

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 2 & & & \\ 1 & 6 & 11 & 6 & & \\ 1 & 10 & 35 & 50 & 24 & \\ 1 & 15 & 85 & 225 & 274 & 120 \end{pmatrix}$$

Note, as mentioned above, that the diagonals of $S$ are the shifted columns of $R$.

Reversed, reduced partition polynomials $RR$, padded A259456:

$$\begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 2 & 3 & & & \\ 0 & 6 & 20 & 15 & & \\ 0 & 24 & 130 & 210 & 105 & \\ 0 & 120 & 924 & 2380 & 2520 & 945 \end{pmatrix}$$

Associated formal h-vectors $H_n(t)$, A112007, for $RR$:

$$\begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 2 & 1 & & & \\ 0 & 6 & 8 & 1 & & \\ 0 & 24 & 58 & 22 & 1 & \\ 0 & 120 & 444 & 328 & 52 & 1 \end{pmatrix},$$

so

$$H_3(t) = (1-t)^4 \frac{1}{1-t} RR_3\left(\frac{t}{1-t}\right) = (1-t)^3 \left(6\frac{t}{1-t} + 20\left(\frac{t}{1-t}\right)^2 + 15\left(\frac{t}{1-t}\right)^3\right)$$

$$= 6t + 8t^2 + t^3,$$

and

$$\frac{H_3(t)}{(1-t)^7} = \frac{6t + 8t^2 + t^3}{(1-t)^7} = 6t + 50t^2 + 225t^4 + \cdots$$
$$= \text{fourth diagonal of S} = \text{shifted fourth column of R}\ .$$

### B) Stirling polynomials of the second kind (Bell / Touchard / Exponential polynomials)

Stirling numbers of the second kind, $S$ matrix, padded A008277, A048993:

$$\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 7 & 6 & 1 \\ 0 & 1 & 15 & 25 & 10 & 1 \\ 0 & 1 & 31 & 90 & 65 & 15 & 1 \end{pmatrix}$$

Revert $R$, A008278, of $S$:

$$\begin{pmatrix} 1 \\ 1 & 1 \\ 1 & 3 & 1 \\ 1 & 6 & 7 & 1 \\ 1 & 10 & 25 & 15 & 1 \\ 1 & 15 & 65 & 90 & 31 & 1 \end{pmatrix}$$

Note, as mentioned above, that the diagonals of $S$ are the shifted columns of $R$.

Reversed, reduced partition polynomials $RR$, padded A134991:

$$\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 3 \\ 0 & 1 & 10 & 15 \\ 0 & 1 & 25 & 105 & 105 \\ 0 & 1 & 56 & 490 & 1260 & 945 \end{pmatrix}$$

Associated formal h-vectors $H_n(t)$, A008517, for $RR$:

$$\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 8 & 6 \\ 0 & 1 & 22 & 58 & 24 \\ 0 & 1 & 52 & 328 & 444 & 120 \end{pmatrix},$$

so

$$H_3(t) = (1-t)^4 \frac{1}{1-t} RR_3\left(\frac{t}{1-t}\right) = (1-t)^3\left(1\frac{t}{1-t} + 10\left(\frac{t}{1-t}\right)^2 + 15\left(\frac{t}{1-t}\right)^3\right)$$

$$= 1t + 8t^2 + 6t^3,$$

and

$$\frac{H_3(t)}{(1-t)^7} = \frac{1t+8t^2+6t^3}{(1-t)^7} = 1t + 15t^2 + 90t^4 + 350t^4 + \cdots$$

$$= \text{fourth diagonal of S} = \text{shifted fourth column of R}\,.$$

## C) Lah polynomials (normalized Laguerre polynomials of the order -1)

Lah numbers, $S$ matrix, padded A105278, unsigned padded A008297, unsigned A111596:

$$\begin{pmatrix} 1 \\ 0 & 1 \\ 0 & 2 & 1 \\ 0 & 6 & 6 & 1 \\ 0 & 24 & 36 & 12 & 1 \\ 0 & 120 & 240 & 120 & 20 & 1 \\ 0 & 720 & 1800 & 1200 & 300 & 30 & 1 \end{pmatrix}$$

Revert $R$, A089231, of $S$:

$$\begin{pmatrix} 1 \\ 1 & 2 \\ 1 & 6 & 6 \\ 1 & 12 & 36 & 24 \\ 1 & 20 & 120 & 240 & 120 \\ 1 & 30 & 300 & 1200 & 1800 & 720 \end{pmatrix},$$

whose columns divided by the first element of the column are the columns of A001263, the Narayana numbers,

$$\begin{pmatrix} 1 & & & & & \\ 1 & 1 & & & & \\ 1 & 3 & 1 & & & \\ 1 & 6 & 6 & 1 & & \\ 1 & 10 & 20 & 10 & 1 & \\ 1 & 15 & 50 & 50 & 15 & 1 \end{pmatrix}$$

Note, as mentioned above, that the diagonals of $S$ are the shifted columns of $R$.

Reversed, reduced partition polynomials $RR$, scaled A086810, scaled, padded A033282:

$$\begin{pmatrix} 1 & & & & & \\ & 2! & & & & \\ & & 3! & & & \\ & & & 4! & & \\ & & & & 5! & \\ & & & & & 6! \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 2 & & & \\ 0 & 1 & 5 & 5 & & \\ 0 & 1 & 9 & 21 & 14 & \\ 0 & 1 & 14 & 56 & 84 & 42 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & & \\ 0 & 2 & & & & \\ 0 & 6 & 12 & & & \\ 0 & 24 & 120 & 120 & & \\ 0 & 120 & 1080 & 2520 & 1680 & \\ 0 & 720 & 10080 & 40320 & 60480 & 30240 \end{pmatrix}$$

Associated formal h-vectors $H_n(t)$, scaled A090181 (scaled,padded A001263, the Narayana numbers) for $RR$:

$$\begin{pmatrix} 1 & & & & & \\ & 2! & & & & \\ & & 3! & & & \\ & & & 4! & & \\ & & & & 5! & \\ & & & & & 6! \end{pmatrix} \begin{pmatrix} 1 & & & & & \\ 0 & 1 & & & & \\ 0 & 1 & 1 & & & \\ 0 & 1 & 3 & 1 & & \\ 0 & 1 & 6 & 6 & 1 & \\ 0 & 1 & 10 & 20 & 10 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & & & & & \\ 0 & 2 & & & & \\ 0 & 6 & 6 & & & \\ 0 & 24 & 72 & 24 & & \\ 0 & 120 & 720 & 720 & 120 & \\ 0 & 720 & 7200 & 14400 & 7200 & 720 \end{pmatrix},$$

so

$$H_3(t) = (1-t)^4 \frac{1}{1-t} RR_3\left(\frac{t}{1-t}\right) = (1-t)^3 4!\left(1\frac{t}{1-t} + 5\left(\frac{t}{1-t}\right)^2 + 5\left(\frac{t}{1-t}\right)^3\right)$$

$$= 4!(1t + 3t^2 + 1t^3),$$

and

$$\frac{H_3(t)}{(1-t)^7} = \frac{4!(1t + 3t^2 + 1t^3)}{(1-t)^7} = 4!(1t + 10t^2 + 50t^4 + 175t^4 + \cdots)$$

$$= 24t + 240t^2 + 12000t^3 + \cdots$$

$$= \text{fourth diagonal of } S = \text{shifted fourth column of } R$$

$$= 4!(\text{fourth diagonal of A001263}) = 4!(\text{shifted fourth column of A001263}).$$