Part I

Raising ops for logarithmic Appell sequences

Define the raising op for the logarithmic Appell sequence \( R_x \) by

\[
e^{tR_x}1 = A(t)x^t = e^{t\psi(x)} = \sum_{n \geq 0} \psi_n(x) \frac{t^n}{n!}
\]

with \( A(0) = 1 \). Then, with \( \frac{d}{dt}|_{t=0}A(t) = D^n_{t=0}A(t) = a_n \), i.e., \( A(t) = e^{a_t} \),

\[
D^n_{t=0} e^{tR_x}1 = R^n_x 1 = \psi_n(x) = (\ln(x) + a.)^n = \sum_{k=0}^{n} \frac{n!}{k!} (\ln(x))^{n-k} a_k,
\]

and clearly

\[
R_x \psi_n(x) = \psi_{n+1}(x).
\]

Acting on the top equation with the shift op \( e^{sD_t}f(t) = f(s+t) \) gives

\[
e^{sD_t}e^{tR_x}1 = e^{(s+t)R_x}1 = e^{sD_t}A(t)x^t = A(s + t) x^{s+t} = e^{(s+t)\psi(x)},
\]

implying the evolution equation (evaluate \( D_s \) at \( s = 0 \))

\[
D_t A(t)x^t = R_x A(t)x^t = R_x e^{tR_x}1
\]

and

\[
e^{sR_x}e^{tR_x}1 = e^{sR_x}A(t)x^t = A(s + t) x^{s+t},
\]

so

\[
e^{tR_x}x^s = x^t \frac{A(s + t)}{A(s)} x^s = x^t \frac{A(xD_x + t)}{A(xD_x)} x^s.
\]
This can be rewritten in terms of the reciprocal \( \hat{A}(s) = \frac{1}{A(s)} \) as a generalized fractional integro-derivative, or shift, operator acting on a particular basis:

\[
e^{t R_x} \frac{x^s}{\hat{A}(s)} = D_{\hat{A}, x}^{-t} \frac{x^s}{\hat{A}(s)} = \frac{x^{s+t}}{A(s+t)}.
\]

Extracting the ops gives

\[
e^{t R_x} = x^t \frac{A(x D_x + t)}{A(x D_x)} = x^t A(\phi(\cdot; x D_x : +t)) = x^t e^{-(1-c_\hat{x}) x D_x};
\]

\[
= D_{\hat{A}, x}^{-t} = x^t \frac{A(x D_x)}{A(x D_x + t)} = x^t \frac{A(\phi(\cdot; x D_x :))}{A(\phi(\cdot; x D_x :) + t)} = x^t e^{-(1-c_\hat{x}) x D_x};
\]

with \( c_n = \frac{A(n+t)}{A(n)} \), \( \hat{c} = \frac{A(n)}{A(n+t)} \), \( \phi(x) \) equal to the Bell / Touchard / exponential polynomials (Stirling polynomials of the second kind, OEIS [A008277]), and by definition : \( x D_x : = x^n D_x^n \). The finite difference expressions incorporating

\[
(1 - b) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} b_k = \nabla_k b_k
\]

follow from the umbral relations

\[
\frac{A(x D_x + t)}{A(x D_x)} f(x) = \sum_{n \geq 0} \frac{A(n+1)}{A(n+1)} f_n \frac{x^n}{n!} = e^{-(1-c_\hat{x}) x D_x} f(x) = f(x - (1-c) x) = f(c x).
\]

(Careful! From the binomial expansion \((1 - b)^n = b_0\), which need not be equal in general to unity; however, it is unity for Appell sequences.) Evaluating the derivative w.r.t. \( t \) at the origin of the exponentiated raising op gives

\[
R_x = \ln(x) + \frac{A'(x D_x)}{A(x D_x)} = \ln(x) + \frac{d}{d x D_x} \ln[A(x D_x)].
\]

The associated lowering operator is defined as

\[
L \psi_n(x) = n \psi_{n-1}(x) = n (\ln(x) + a.)^{n-1} = x D_x (\ln(x) + a.)^n = x D_x \psi_n(x),
\]

so the commutator for the operators (acting on any \( \psi_n(x) \)) is

\[
[L_x, R_x] = L_x R_x - R_x L_x = \text{Identity}.
\]

And, from the properties of the Pincherle derivative,

\[
\ln(x) + [\ln(A(L_x)), R_x] = \ln(x) + \frac{d}{d L_x} \ln(A(L_x)) = R_x.
\]

Note also that

\[
e^{t R_x} e^{s R_x} = e^{(t+s) R_x}
\]
implies
\[
x^t \frac{A(t + xD_x)}{A(xD_x)} x^s \frac{A(s + xD_x)}{A(xD_x)} = x^{t+s} \frac{A(t + xD_x)}{A(xD_x)} x^t \frac{A(t + xD_x)}{A(xD_x)}
\]

and the same relations for the operator \( x^t A(xD_x) A(t + xD_x) = x^t \frac{A(t + xD_x)}{A(xD_x)} \).
For example, with \( A(x) = x! \), this implies
\[
x^t \left( t + xD_x \right) x^s \left( s + xD_x \right) = x^{t+s} \left( t + s + xD_x \right).
\]

**Part II**

**Recursion relations and integral reps for the raising op**

The raising op can be expressed several ways as in the entry Bernoulli Appells here. The most convenient for binomial relations is, with a change variables to \( \ln(x) = z \),
\[
R_z = z + \frac{d}{dD_z} \ln[M(D_z)] = z + \frac{d}{dD_z} \ln[\exp^mD_z] = z + \frac{d}{dD_z} \exp^{c.D_z} = z + c.\exp^{c.D_z}
\]
with \( c_n \) regarded as the formal cumulants and \( m_n \) as the formal moments of OEIS [A127671] defined by
\[
e^{c.t} = \ln[\exp^m.t] = \ln[M(t)]
\]
with \( M(0) = 1 \). Then
\[
R^n_z 1 = (z + m.)^n = p_n(z),
\]
and a recursion relation follows from the Appell binomial property \( p_n(z + h) = (z + m. + h)^n = (p(z) + h)^n \) (as in the Appell Polynomials, Cumulants, ... entry):
\[
R_z p_n(z) = p_{n+1}(z) = (z + c.\exp^{c.D_z}) p_n(z) = z p_n(z) + c. p_n(c. + z)
\]
\[
= z p_n(z) + c. (c. + p_.(z))^n = z p_n(z) + \sum_{k=0}^n \binom{n}{k} c_{n+1-k} p_k(z)
\]
\[
= (z + c_1) p_n(z) + \sum_{k=1}^n \binom{n}{k} c_{k+1} p_{n-k}(z).
\]
These polynomials are precisely the general Bell polynomials
\[
p_n(z) = B_n(z + c_1, c_2, ..., c_n) = B_n(p_1(z), c_2, ..., c_n) = B_n(x[1], ..., x[n]),
\]
or partition polynomials for the refined Stirling numbers of the second kind, of OEIS A036040 which are an Appell sequence in the distinguished indeterminate \( c_1 \), i.e.,
\[
\frac{d}{dc_1} B_n(c_1, ..., c_n) = (n-1) B_{n-1}(c_1, ..., c_{n-1}) .
\]
They can also be expressed as the cycle index polynomials of the symmetric groups, or the partition polynomials for the refined Stirling numbers of the first kind, of A036039

\[
p_n(z) = CIP_n(z + b_1, b_2, ..., b_n) = CIP_n(p_1(z), b_2, ..., b_n) = CIP_n(x[1], ..., x[n]) , \]

with \( x[1] = b_1 = p_1(z) = z + c_1 \) and \( x[n] = b_n = c_n/(n-1)! \) for \( n > 0 \) with raising op \( R = \frac{b_1}{1-b_1 \cdot D_{b_1}} = b_1 + \sum_{n \geq 1} b_{n+1} D_{b_1}^n \), which are an Appell sequence in the distinguished indeterminate \( b_1 \), i.e.,
\[
\frac{d}{db_1} CIP_n(b_1, ..., b_n) = (n-1) CIP_{n-1}(b_1, ..., b_{n-1}) .
\]

The recursion relation leads to an integral representation for the raising op.

Compare
\[
p_n(z) - (p_1(z) - \omega)^n = p_n(z) - \sum_{k=0}^{n} (-1)^k \binom{n}{k} p_{n-k}(z) \omega^k = \sum_{k=1}^{n} (-1)^{k+1} \binom{n}{k} \omega^k p_{n-k}(z)
\]
with the recursion relation. If we can find a distribution \( \mu(\omega) \) such that its moments are the cumulants \( c_{k+1} \) for \( k > 0 \), we have our integral rep

\[
c_{k+1} = \int (-1)^{k+1} \omega^k \mu(\omega) \, d\omega = -\frac{d^k}{d\alpha^k} \int (e^{-\omega \alpha} - 1) \mu(\omega) \, d\omega|_{\alpha=0},
\]
or
\[
c_{k+1} = \int (e^{c_\alpha} - 1) = -\int (e^{-\omega \alpha} - 1) \mu(\omega) \, d\omega.
\]

If this is essentially a Laplace transform, then the inverse Laplace transform in an appropriate region of evaluation gives \( \mu(\omega) \). Under suitable conditions, the coefficients can be regarded as discrete samples

\[
c_{k+1} = (-1)^{k+1} k! C(k+1)
\]
of a Mellin transform of the distribution:

\[
C(s) = \int_0^\infty \mu(\omega) \frac{\omega^{s-1}}{(s-1)!} \, d\omega .
\]
The distribution may need to be modified (the integral regularized) over strips of \( s \) to obtain an invariant \( C(s) \) over the Mellin dual space, just as for \( C(s) = 1 \).
Since
\[ p_n(z) - (p_n(z) - \omega)^n = p_n(z) - p_n(z - \omega), \]
the recursion relation can be expressed as
\[ Rzp_n(z) = p_{n+1}(z) = (z + c_1) p_n(z) + \sum_{k=1}^{n} \binom{n}{k} c_{k+1} p_{n-k}(z). \]

Or, with the change of variables \( \omega = z - t \),
\[ Rzp_n(z) = p_{n+1}(z) = (z + c_1) p_n(z) + \int_{z-l_2}^{z-l_1} [p_n(z) - p_n(z) - \omega] \mu(\omega) d\omega. \]

Now let
\[ z = \ln(x), \quad t = \ln(v), \quad \text{and} \quad p_n(z) = p_n(\ln(x)) = \psi_n(x), \]
then
\[ Rx\psi_n(x) = \psi_{n+1}(x) = (\ln(x) + c_1) \psi_n(x) + \int_{x e^{-t_1}}^{x e^{-t_2}} [\psi_n(x) - \psi_n(v)] \mu(\ln(x/v)) \frac{1}{v} dv. \]

So, for functions analytic about the origin,
\[ Rz f(z) = (z + c_1) f(z) + \int_{z-l_2}^{z-l_1} [f(z) - f(t)] \mu(z-t) dt, \]
and
\[ Rz g(x) = (\ln(x) + c_1) g(x) + \int_{x e^{-t_1}}^{x e^{-t_2}} [g(x) - g(t)] \mu(\ln(x/t)) \frac{1}{t} dt. \]

**EXAMPLES:**

1) As gleaned from the Riemann zeta Appell sequence of the MathOverflow question “Riemann zeta function at positive integers and an Appell sequence of polynomials related to fractional calculus” and the entry here “On the Mellin interpolation of differential ops and associated inﬁnigens and Appell polynomials ...”, the distributions for the Riemann zeta Appell sequences are signed and unsigned
\[ \mu(\omega) = \frac{1}{e^\omega - 1}, \]
and the cumulants (within an overall sign) for \( n > 1 \) are
\[ c_n = (-1)^n (n-1)! \zeta(n) = (-1)^n \int_0^\infty \omega^{n-1} \frac{1}{e^\omega - 1} d\omega. \]
with
\[c. (e^{c\cdot \alpha} - 1) = \sum_{n \geq 1} (-1)^{n+1} \zeta(n+1) \alpha^n = \Psi(1 + \alpha) + \gamma\]
\[= \frac{d}{d\alpha} e^{c\cdot \alpha} - c_1 = \frac{d}{d\alpha} \log [e^{m\cdot \alpha}] - c_1 = \frac{d}{d\alpha} \log [\alpha] + \gamma\]

for \(|\alpha| < 1\), where \(\Psi\) is the digamma, or Psi, function with \(c_1 = \frac{d}{d\alpha} \log (\alpha)!\) at \(\alpha = 0 = -\gamma = \Psi(1) = -0.577\ldots\), the negated Euler-Mascheroni constant. Also, \(l_2 = \infty\), \(l_1 = 0\), and

\[R_z f(z) = (z - \gamma) f(z) + \int_{-\infty}^{z} \left[ f(z) - f(t) \right] \frac{1}{e^{z-t} - 1} dt,\]
and

\[R_z g(x) = (\ln(x) - \gamma) g(x) + \int_{0}^{x} \frac{g(x) - g(t)}{x - t} dt\]
\[= 2 \ln(x) g(x) + \frac{1}{2\pi i} \int_{|z-x|=\varepsilon} \frac{-\ln(z-x) - \gamma}{z-x} g(z) dz\]
\[= 2 \ln(x) g(x) + \frac{1}{2\pi} \int_{-\pi}^{\pi} (-i\theta - \ln(x) - \gamma) g(x(1 + e^{i\theta})) d\theta.\]

Specifically, for this choice of overall sign for the cumulants, we have obtained the raising operator or infingen associated to the shifted Laguerre operator \(xD_x x\), discussed in the earlier entry:

\[R_x x^{-s} = (\ln(x) - \gamma) x^{-s} + \int_{0}^{x} \frac{x^{-s} - t^{-s}}{x - t} dt = R_x x^{-s} = [(\ln(x) - \gamma) + \int_{0}^{1} \frac{1 - t^{-s}}{1 - t} dt] x^{-s}\]
\[= [\ln(x) + \Psi(1-s)] x^{-s} = [\ln(x) + \Psi(1+xD_x)] x^{-s}\]
\[= \ln(x D_x x) x^{-s} = [2 \ln(x) + \ln(D_x)] x^{-s}.\]

Then acting on functions analytic at the origin,

\[R_x = \ln(x) + \Psi(1 + xD_x)\]
\[= \ln(x) - \gamma + \sum_{n \geq 0} \left( \frac{1}{n+1} - \frac{1}{n+1 + xD_x} \right)\]
\[= \ln(x) - \gamma + \int_{0}^{1} \frac{1 - t x D_x}{1 - t} dt\]
\[= \ln(x) - \gamma + \sum_{n \geq 1} H_n \frac{x^n D_x^n}{n!}\]
\[ R_x f(x) = \ln(x) - \gamma + \sum_{n \geq 1} (-1)^n \left( \frac{1}{n} \right) x^n D_n^x \]

where \( b_0 = H_0 = 0 \) (note that \( (1 - b_0)^n = b_0 = 0 \), in this case, and \( b_n = -1/n \) otherwise) and the harmonic numbers \( H_n \) are defined for \( n > 0 \) by

\[ H_n = \sum_{k=1}^{n} \frac{1}{k} = \int_0^1 \frac{1 - u^n}{1 - u} du = \gamma + \Psi(n + 1) \]

so

\[ R_x f(x) = \ln(x) - \gamma + (1 - b_1) x D_1^x f(x) \]

\[ = (\ln(x) - \gamma) f(x) + \sum_{n \geq 1} H_n f^{(n)}(0) \frac{x^n}{n!} = (\ln(x) - \gamma) f(x) + \int_0^x \frac{f(x) - f(t)}{x - t} dt . \]

The operator can also be expressed in terms of the entire function exponential integral

\[ Ein(z) = \int_0^z \frac{1 - e^{-t}}{t} dt = \sum_{n \geq 1} (-1)^n \left( \frac{-1}{n} \right) \frac{z^n}{n!}, \]

with \( x D_x := x^n D_n^x \) by definition, as

\[ R_x = \ln(x) - \gamma + Ein(\; x D_x \;). \]

2) For the Bernoulli polynomials discussed in the Bernoulli Appells entry, the raising op for the logarithmic Appell sequence is

\[ R_x = \ln(x) + \sum_{n \geq 0} (-1)^n \frac{B_{n+1}}{n+1} \frac{(xD_x)^n}{n!} \]

\[ = \ln(x) - \frac{1}{2} + \sum_{n \geq 1} (-1)^n 2 \frac{\zeta(2n)}{(2\pi)^{2n}} (xD_x)^{2n-1} \]

\[ = \ln(x) - 1 + \frac{1}{xD_x} - \frac{1}{e^{xD_x} - 1} \]
We then obtain, for $t$ about the origin,

$$e^{tR_{x} s} = \frac{A(s + t) x^{s + t}}{A(s)} = \frac{s + t - \frac{1}{s}}{e^{s + t} - 1} x^{s + t} = e^{\ln(x) - \frac{1}{2} \frac{i x D_{x}}{x D_{x}}} \frac{\sinh \left( \frac{i x D_{x}}{2 x D_{x}} \right)}{\sinh \left( \frac{i x D_{x} + i \pi}{2 x D_{x}} \right)} x^{s} = x^{t} e^{-\frac{1}{2} \frac{i \frac{x D_{x}}{x D_{x}}}{\sin \left( \frac{i \frac{x D_{x}}{x D_{x}}}{i \frac{i \frac{x D_{x}}{x D_{x}}}{i \frac{\pi}{2}}} \right)}} \left( \frac{i \frac{x D_{x}}{x D_{x}}}{i \frac{i \frac{x D_{x}}{x D_{x}}}{2 \pi}} \right)^{2} x^{s} = x^{t} e^{-\frac{1}{2} \left( \frac{1}{2} \pi \right)^{2} \frac{1}{2} \frac{x D_{x}}{x D_{x}}} x^{s} = x^{t} e^{-\frac{1}{2} \left( x^{\frac{2 \pi}{\pi} - 1} \right)^{x D_{x}}} x^{s} = x^{t} e^{-\frac{1}{2} \left( x^{\frac{2 \pi}{\pi} - 1} \right)^{x D_{x}}} x^{s}.
$$

In the last equality, the last factor from the left in the modulus transforms $x^{s}$ to $x^{t} x^{s}$ and the first factor reverses the transformation as

$$(x^{q-1}) x^{s} = x^{s} (x^{q-1})^{s} = x^{s} = \sum_{n \geq 0} (-1)^{n} (1 - x^{q-1})^{n} \frac{x D_{x}^{n}}{n!} x^{s}.$$
(Recall that by definition: \( AB : n = A^n B^n \) so that: \( xD_x : n = x^n D^n \) and : \( D_x x : n = D^n x^n \).)

The middle factor is a generalized Laguerre operator giving

\[
D_{x}x : \frac{t}{\pi} x^{\frac{t}{\pi}} = D_{x}^\frac{t}{\pi} x^{\frac{t}{\pi}} x^{\frac{t}{\pi}} = \left( \frac{t + s}{2\pi} \right)! D_{x}^{\frac{t}{2\pi}} \frac{x^{\frac{t}{2\pi}}}{\left( \frac{t + s}{2\pi} \right)!} = \left( \frac{t + s}{2\pi} \right)! x^{\frac{t}{2\pi}}
\]

with one convolution rep of the fractional integro-derivative being the Hadamard finite part of

\[
D_{x}^{\frac{t}{\pi}} x^{\frac{t}{\pi}} = \int_{0}^{x} \frac{(x - u)^{-i\frac{t}{\pi} - 1}}{(-i\frac{t}{\pi} - 1)!} u^{i\frac{t}{\pi} - 1} du = \int_{0}^{x} \frac{(x - u)^{i\frac{t}{\pi} - 1}}{(-i\frac{t}{\pi} - 1)!} (x - u)^{i\frac{t}{\pi} - 1} du
\]

\[
= x^{i\frac{t}{\pi}} \int_{0}^{1} \frac{u^{-i\frac{t}{\pi} - 1}}{(-i\frac{t}{\pi} - 1)!} (1 - u)^{i\frac{t}{\pi} - 1} du = x^{i\frac{t}{\pi}} \left( \frac{t}{2\pi} \right)! \sum_{n \geq 0} \left( \frac{i\frac{t}{\pi}}{n} \right) \sin(\pi(n - i\frac{t}{\pi})) \frac{1}{\pi(n - i\frac{t}{\pi})}
\]

\[
= \left( \frac{t}{2\pi} \right)! \sin(\pi(n - i\frac{t}{\pi})) \frac{1}{i\frac{t}{\pi}} x^{i\frac{t}{\pi}},
\]

with the summation valid for \( \text{Real} \left( i\frac{t}{\pi} \right) > -1 \). Other reps are given in other notes at this site and are reflected in the integral reps above. The last expression of the exponentiated raising op may also be expanded out and the middle two factors reduced as

\[
e^{iR_x x^{\sigma}} = x^{\frac{t}{2\pi}} \left( \frac{t}{2\pi} - 1 \right)^{i\frac{t}{\pi}} \cdot D_{x}x : \frac{t}{\pi} x^{(x - 2)\frac{t}{\pi}D_{x}} : D_{x}x : \frac{t}{\pi} \left( \frac{x - t}{\pi} - 1 \right)^{i\frac{t}{\pi}}
\]

**Part III**

**Convolution rep for** \( D_{A_{x}}^{-\frac{t}{\pi}} = \exp(t R_{x}) \) **from the Mellin transform**

Acting on a function representable as an inverse Mellin transform,

\[
e^{iR_{x}} f(x) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{f}_{M}(s) e^{iR_{x} x^{-s}} ds = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{f}_{M}(s) x^{t} \frac{A(xD_{x} + t)}{A(xD_{x})} x^{-s} ds
\]

\[
= \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{f}_{M}(s) \frac{A(-s + t)}{A(-s)} x^{-s+t} ds
\]

\[
= x^{t} \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \tilde{f}_{M}(s) \hat{K}_{M}(1 - s; t) x^{-s} ds
\]

with

\[
\hat{K}_{M}(s; t) = \frac{A(s + t - 1)}{A(s - 1)} = \frac{\hat{A}(s - 1)}{A(s + t - 1)}.
\]
Then from the Mellin convolution theorem,

\[ e^{tR_x} f(x) = x^t \int_0^\infty K(u; t) f(xu) du = x^t \int_0^\infty \frac{1}{x} K \left( \frac{u}{x}; t \right) f(u) du \]

\[ = x^t \sum_{n \geq 0} \frac{A(n + t)}{A(n)} f_n \frac{x^n}{n!} = x^t e^{c \cdot x D_x = 0} f(x) = x^t e^{-(1-c) \cdot x D_x} f(x) \]

where \( K(x; t) \) is the inverse Mellin transform of \( \tilde{K}_M(s; t) \) and \( f_n = D^n_{x=0} f(x) \).

For example, let \( A(s) = 1/s! \), corresponding to the infinitesimal \( R_x = \ln(x) - \Psi(1 + xD_x) = -\ln(D_x) \) for the fractional integro-derivatives \( D_x^{-\alpha} \). Then for \( \sigma > 0 \) and \( t \) non-integral,

\[ K(x; t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \tilde{K}_M(s; t) x^{-s} ds = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \frac{(s-1)!}{(s+t-1)!} x^{-s} ds \]

\[ = H(1-x) \sum_{n \geq 0} (-1)^n \frac{1}{(t-1-n)!n!} x^n = H(1-x) \frac{(1-x)^{t-1}}{(t-1)!}, \]

where \( H(x) \) is the Heaviside step function, so

\[ e^{\beta R_x} \frac{x^\alpha}{\alpha!} = D^{-\beta} \frac{x^\alpha}{\alpha!} = \frac{x^{\alpha+\beta}}{(\alpha+\beta)!} \]

\[ = x^\beta \int_0^\infty \frac{1}{x} K \left( \frac{u}{x}; \beta \right) \frac{u^\alpha}{\alpha!} du = x^\beta \int_0^1 \frac{1}{x} \frac{(1-x)^{\beta-1}}{(\beta-1)!} \frac{u^\alpha}{\alpha!} du \]

\[ = \int_0^x (x-u)^{\beta-1} \frac{u^\alpha}{\alpha!} du. \]

As another illustration note that if \( A(x) = 1 \), then \( \tilde{K}_M(s; t) = 1 \), and the inverse Mellin transform gives \( K(x; t) = \delta(1-x) \); therefore, \( D_x^{-\alpha} = x^\alpha \), consistent with \( R_x = \ln(x) \) and \( e^{tR_x} = x^t \).