1 The Elliptic Lie Triad

The basic Lie Triad is a pair of functions analytic and null-valued at the origin and mutually inverse under composition together with a generator formed from the reciprocal of the derivative of one of the functions, a generator from which two infinitesimal generators (innigens) emerge: the Lie innigen (i.e., ‘velocity/tangent’ vector or derivative) whose action when exponentiated can generate one of the pair of inverse functions and the integral innigen (or differential form) that when integrated along a path from the origin gives the other of the pair.

The general ordered Lie triad is

$$\text{Triad} = [h(x), h^{-1}(x), g(x) = 1/(dh^{-1}(x)/dx)] .$$

(There, of course, is a dual reciprocal triad formed from switching the pair of inverse functions in the triad.) We are going to look at the KdV and Riccati equations, formal group law, and elliptic curve attached to the elliptic Lie triad $\text{Triad}_E$ defined by the quadratic elliptic generator

$$g_E(x) = e_0 + e_1 x + e_2 x^2$$

associated with the vectorial innigen (reflecting the inverse function theorem/chain rule)

$$g(x) \frac{d}{dx} = \frac{d}{dh^{-1}(x)} = h'(h^{-1}(x)) \frac{d}{dx}$$

and differential form

$$\frac{dx}{g(x)} = dh^{-1}(x) ,$$

then specialize to the hyperbolic tangent triad, associated with the normalized Bernoulli and Euler numbers,

$$\text{Triad}_{HT} = [\tanh(x), \text{atanh}(x), 1 - x^2] .$$
The KdV and Riccati Equations for the Elliptic Lie Triad

The general KdV (Korteweg-de Vries) equation is

\[ u_t - u \cdot u_x + \epsilon u_{xxx} = 0 , \]

which may be re-expressed as

\[ k_2 \bar{u}_t - k_1 \bar{u} \cdot \bar{u}_x + \bar{\epsilon} \bar{u}_{\bar{x}\bar{x}} = 0 , \]

with \( u(x,t) = \bar{u}(\bar{x},\bar{t}) \), \( k_1 x = \bar{x} \), and \( k_2 t = \bar{t} \), so we need deal only with the more simply expressed KdV equation. Following the arguments in “Bernoulli numbers and solitons – revisited” by Rzadkowski to relate the KdV equation to a Riccati equation and the solution of the standard autonomous ODE, we assume a travelling wave solution

\[ u(x,t) = f(x - ct) . \]

and the KdV equation becomes

\[ cf' + ff' = \epsilon f''' \]

\[ = \frac{d}{dx} (cf + \frac{1}{2} f^2 + A) = \frac{d}{dx} \epsilon f'' \]

so

\[ f'(cf + \frac{1}{2} f^2 + A) = f' \epsilon f'' \]

\[ = \frac{d}{dx} \left( \frac{1}{6} f^3 + \frac{c}{2} f^2 + Af \right) = \frac{\epsilon}{2} \frac{d}{dx} (f')^2 , \]

implying

\[ \frac{\epsilon}{2} (f')^2 = \frac{1}{6} f^3 + \frac{c}{2} f^2 + Af + B . \quad [\text{Eqn. *]} \]

Now let \( A = B = 0 \) and define \( f \) in terms of the solution of the standard autonomous ODE for the integral curves of a vector field

\[ f(x) = \frac{dh(x)}{dx} = g(h(x)) . \]

Then

\[ f'(x) = g'(h(x)) h'(x) = g'(h) f , \]

and substituting into the cubic equation Eqn. *,

\[ \frac{\epsilon}{2} (g'(h) f)^2 = \frac{1}{6} f^3 + \frac{c}{2} f^2 , \]
or
\[ \frac{\epsilon}{2} (g'(h))^2 = \frac{1}{6} f + \frac{c}{2} = \frac{1}{6} g(h) + \frac{c}{2}. \]

Now specialize to the vector field defined by the quadratic elliptic generator
\[ g_E(x) = e_2 x^2 + e_1 x + e_0, \]
and substitute this into the previous equation, giving
\[ \frac{\epsilon}{2} (2e_2 x + e_1)^2 = \frac{1}{6} (e_2 x^2 + e_1 x + e_0) + \frac{c}{2} \]
\[ = \frac{\epsilon}{2} (4e_2^2 x^2 + 4e_2 e_1 x + e_1^2), \]
identifying
\[ \epsilon = \frac{1}{12e_2} \quad \text{and} \quad c = \frac{e_1^2}{12e_2} - \frac{e_0}{3}. \]

Reprising then, the Riccati equation associated with the vector field is defined by the associated autonomous ODE for the integral curves
\[ h'(\omega) = g(h(\omega)) = e_0 + e_1 h(\omega) + e_2 h^2(\omega), \]
or
\[ z'(\omega) = g(z(\omega)) = e_0 + e_1 z(\omega) + e_2 z^2(\omega) \]
with the associated elliptic Lie infinigen in \( sl(2) \)
\[ g_E(z) \frac{d}{dz} = e_0 \frac{d}{dz} + e_1 z(\omega) \frac{d}{dz} + e_2 z^2(\omega) \frac{d}{dz} \]
and associated KdV
\[ u_t - u \cdot u_x + \frac{1}{12e_2} u_{xxx} = 0 \]
with the travelling soliton solution
\[ u(x,t) = f(x - ct) = h'(x - ct) = g(h(x - ct)) = e_0 + e_1 h(x - ct) + e_2 h^2(x - ct), \]
where
\[ \epsilon = \frac{e_1^2}{12e_2} - \frac{e_0}{3}. \]

In the next section, the infinigens are used to generate the pair of inverse functions in the elliptic triad.
The Compositional Inverse Pair for the Elliptic Lie Triad

The elliptic triad by construction contains the compositional inverse pair of functions \( \omega = h(z) \) and \( z = h^{-1}(\omega) \) analytic and null-valued at the origin defined by

\[
g(\omega) = \frac{1}{dh^{-1}(\omega)} = e_0 + e_1 \omega + e_2 \omega^2,
\]

so integrating the differential form along a path from the origin

\[
z(\omega) = h^{-1}(\omega) = \int_0^\omega dh^{-1}(\omega) = \int_0^\omega \frac{1}{g(\omega)} = \int_0^\omega \frac{1}{e_2 \omega^2 + e_1 \omega + e_0} d\omega
\]

\[
= \int_0^\omega \frac{1}{e_2(\omega - \omega_1)(\omega - \omega_2)} d\omega = \frac{1}{e_2(\omega_2 - \omega_1)} \int_0^\omega \left[ \frac{1}{\omega - \omega_2} - \frac{1}{\omega - \omega_1} \right] d\omega
\]

\[
= \frac{1}{e_2(\omega_2 - \omega_1)} \ln[(1 - \frac{\omega}{\omega_2})/(1 - \frac{\omega}{\omega_1})].
\]

(You must be careful about defining branch cuts for specific values of the parameters.) The inverse is given in OEIS A008292 with \( a = -1/\omega_2 \) and \( b = -1/\omega_1 \) as

\[
\omega = h(z) = \frac{e^{-e_2 \omega_2 z} - e^{-e_2 \omega_1 z}}{e^{-e_2 \omega_2 z}/\omega_2 - e^{-e_2 \omega_1 z}/\omega_1}.
\]

The Elliptic Formal Group Law and Elliptic Curve

Exponentiating the vectorial innigen provides insights into combinatorics and geometry underlying the formal group law:

\[
\exp[x g(\omega) \frac{d}{d\omega}] \omega = \exp[x \frac{d}{dh^{-1}(\omega)}] \omega = \exp[x \frac{d}{dz}] h(z) = h(x + z) = h[x + h^{-1}(\omega)].
\]

Then evaluating at \( \omega = 0 \) gives \( h(x) \). The connections to a local Lie group, the geometry underlying this relation, and the refined Eulerian numbers for compositional inversion for general \( g(x) \) can be found at OEIS A145271. The specific elliptic generator \( g_E(x) \) is related to the bivariate e.g.f. for the unrefined Eulerian numbers [A008292] and [A123125], which are rife with connections to the Coxeter groups \( A_n \) and \( B_n \). To relate the compositional inversion to the Stasheff polytopes, or associahedra, and other combinatorial-geometric constructs, see the cross-references at A145271 for the different types of inversion formulas.
The $FGL_E$ evaluates simply to

$$FGL_E(x, y) = h[h^{-1}(x) + h^{-1}(y)] = \frac{x + y - (\omega_1 + \omega_2) \frac{xy}{\omega_1 \omega_2}}{1 - \frac{xy}{\omega_1 \omega_2}}.$$

The relation to elliptic curves of the Jacobi type can be seen from the path integral of the differential form given above:

$$z(\omega) = h^{-1}(\omega) = \int_0^\omega \frac{1}{e_2 \omega^2 + e_1 \omega + e_0} d\omega = \int_0^\omega \frac{1}{\sqrt{(e_2 \omega^2 + e_1 \omega + e_0)^2}} d\omega$$

$$= \int_0^\omega \frac{1}{\sqrt{e_2 (\omega - \omega_2)(\omega - \omega_1)^2}} d\omega.$$

5 Summarizing the Characteristics of the Elliptic Lie Triad

Reprising, the elliptic Lie triad is

$$Triad_E[h(x), h^{-1}(x), g_E(x)]$$

with associated

1) Generator:

$$g_E(x) = e_0 + e_1 x + e_2 x^2 = e_2 (x - \omega_1)(x - \omega_2) = \frac{1}{d[h^{-1}(x)]/dx} = h'(h^{-1}(x))$$

with $e_0 = e_2 \omega_1 \omega_2$ and $e_1 = -e_2(\omega_1 + \omega_2)$

2) Inverse function:

$$h^{-1}(\omega) = \frac{1}{e_2(\omega_2 - \omega_1)} \ln[(1 - \frac{\omega}{\omega_2})/(1 - \frac{\omega}{\omega_1})]$$

3) Forward function:
\[ h(x) = \frac{e^{-e_2x^2} - e^{-e_2x^2}}{e^{-e_2x^2} - e^{-e_2x^2}} \]

4) Riccati equation:

\[ \frac{d}{d\omega} h(\omega) = g(h(\omega)) = e_0 + e_1 h(\omega) + e_2 h^2(\omega) \]

5) KdV equation:

\[ u_t - u \cdot u_x + \frac{1}{12\xi_2} u_{xxx} = 0 \]

6) KdV soliton:

\[ u(x,t) = h'(x - ct) = g(h(x - ct)) = e_0 + e_1 h(x - ct) + e_2 h^2(x - ct) \]

7) Soliton velocity:

\[ c = \frac{e_1^2}{12\xi_2} - \frac{e_0}{3} = \frac{e_1^2 - 4e_0e_2}{12\xi_2} = \frac{e_2}{3} \left( \frac{\omega_2 - \omega_1}{2} \right)^2 = \frac{D^3 u(x,t) - 6\xi_2 D_x u(x,t)}{12\xi_2 D_x u(x,t)} \]

\[ = \frac{1}{6\xi_2} S_x \{ h^{-1}(x) \} (g_E(x))^2 = -\frac{1}{6\xi_2} S_x \{ h(x) \} = -\frac{1}{6\xi_2} \{ D^2_x \ln[u(x,t)] - \frac{1}{2} [D_x \ln(u(x,t))]^2 \}, \]

where \( S_x \) is the Schwarzian derivative defined in Section 8. (See also the Michor reference, pages 55-56.)

8) Formal group law:

\[ FGL_E(x,y) = h[h^{-1}(x) + h^{-1}(y)] = \frac{x + y - (\omega_1 + \omega_2) \cdot \frac{x}{e_1} - \frac{y}{e_2}}{1 - \frac{x}{e_1} - \frac{y}{e_2}} = \frac{x + y + \left( \frac{e_2}{e_1} \right)xy}{1 - \left( \frac{e_2}{e_1} \right)xy} \]
6 Degeneracy: Confluence of zeroes of the generator and Moebius transformations

When \( \omega_1 = \omega_2 \), the travelling wave solution degenerates into a stationary solution with zero velocity, associated with an inverse pair of rational functions related to projective space and the Moebius, or linear fractional, transformations that fail to vanish, in general, at the origin—a homogeneous solution to the KdV equation, which have vanishing Schwarzian.

1) Generator:

\[
g_E(x) = e_0 + e_1 x + e_2 x^2 = e_2 (x - \omega_1)^2 = \frac{1}{d[h^{-1}(x)]/dx} = h'(h^{-1}(x))
\]

with \( e_0 = e_2 \omega_1^2 \) and \( e_1 = -2e_2 \omega_1 \)

2) Inverse function:

\[
h^{-1}(x) = \frac{ax + b}{e_2 x - e_2 \omega_1} \quad \text{with} \quad b = -(\omega_1 a + 1)
\]

(Note that we are free to choose \( b = 0 \) as a special solution, giving \( h(0) = h^{-1}(0) = 0 \).)

3) Forward function:

\[
h(x) = \frac{-e_2 \omega_1 x - b}{-e_2 x + a}
\]

4) Riccati equation:

\[
\frac{d}{d\omega} h(\omega) = g(h(\omega)) = e_0 + e_1 h(\omega) + e_2 h^2(\omega)
\]

5) Degenerate KdV equation:
\[-u \cdot u_x + \frac{1}{12 e_2} u_{xxx} = 0 = -\frac{1}{2}u^2(x) + \frac{1}{12 e_2} u_{xx} \]

6) Degenerate KdV solution:

\[
u(x) = h'(x) = g(h(x)) = e_0 + e_1 h(x) + e_2 h^2(x) = \frac{e_2(\omega_1 a + b)}{(-e_2 x + a)^2} \]

7) Soliton velocity:

\[
\begin{align*}
    c &= 0 = \frac{D^3 u(x,t) - 6e_2 D_x u(x,t)}{12e_2 D_x u(x,t)} = \frac{1}{6e_2} S_x \{h^{-1}(x)\} (g_E(x))^2 = -\frac{1}{6e_2} S_x \{h(x)\} \\
    &= -\frac{1}{6e_2} \{D^2 \ln[u(x,t)] - \frac{1}{2} [D_x \ln(u(x,t))]^2 \},
\end{align*}
\]

8) Formal group law:

With \( a = -\frac{1}{\omega_1} \) so that \( b = 0 \) and \( h(0) = 0 = h^{-1}(0) \),

\[
FGL_E(x, y) = h[h^{-1}(x) + h^{-1}(y)] = \frac{x+y-\frac{2}{\omega_1} xy}{1-\frac{2}{\omega_1} xy} = \frac{x+y+(\frac{2}{\omega_1})xy}{1-(\frac{2}{\omega_1})xy}
\]

7 Specializing to the Hyperbolic Tangent Triad

Taking \( e_0 = 1, e_1 = 0, \) and \( e_2 = -1 \) in the generator for the elliptic triad specializes it to the hyperbolic tangent triad

\[
Triad_{HT} = [\tanh(x), \text{atanh}(x), 1 - x^2],
\]

with associated

1) KdV equation: \( u_t - u \cdot u_x - \frac{1}{12} u_{xxx} = 0 \)
2) Riccati equation: 
\[ h'(\omega) = 1 - h^2(\omega) = z'(\omega) = 1 - z^2(\omega) \]

3) Formal group law: 
\[ FGL_{HT}(x, y) = h[h^{-1}(x) + h^{-1}(y)] = \frac{x + y}{1 + xy} \]

4) KdV soliton: 
\[ \frac{\partial}{\partial x} \left[ \tanh(x + t/3) \right] = 1 - \tanh^2(x + t/3). \]

The \( FGL_{HT} \) gives the addition law for collinear velocities in special relativity:

\[ h(x + y) = \frac{h(x) + h(y)}{1 + h(x) h(y)} = \tanh(x + y) = \frac{\tanh(x) + \tanh(y)}{1 + \tanh(x) \tanh(y)}. \]

The hyperbolic tangent can be regarded as an exponential generating function for the number of connected components in the space of M-polynomials in hyperbolic functions (from OEIS A000111) or for a proportionality factor in the Kervaire-Milnor formula in homotopy theory for hyper-spheres involving normalized Bernoulli numbers. There is more on this particular triad given in the bottom notes of the entry here on the Kervaire-Milnor formula. Note that the HT infinitesimal appears in the differential equation characterizing the Legendre polynomials (cf. also MO-Q: Geometric picture of invariant differential of an elliptic curve):

\[ [(1 - x^2)D_x]^2 L_n(x) + (n + 1)n (1 - x^2)D_x L_n(x) = 0, \]

whose ordinary generating function is

\[ \sum_{n \geq 0} L_n(u)x^n = \frac{1}{\sqrt{1 - ux + x^2}}. \]

8 Relations to Appell sequences: the Swiss-knife and Grassmann polynomials

The Swiss-knife polynomials \( K_n(x) \) (OEIS A119879) are an Appell sequence of polynomials that can be used to calculate the Bernoulli, Genocchi, Euler, tangent, and Springer numbers and are related to a quantum algebra as developed by Sukumar and Hodges (see also my answer to the related MO-Q on the e.g.f. of the Bernoulli numbers). The sequence has the lowering and raising ops

\[ L = d/dx = D_x \quad \text{and} \quad R_{HT} = x - \tanh(D_x), \]
satisfying
\[ L K_n(x) = K_{n-1}(x) \quad \text{and} \quad R_{HT} K_n(x) = K_{n+1}(x) \; ; \]
consequently, the e.g.f. of the polynomials satisfies
\[ e^t R_{HT} 1 = e^{K.(x)t} = \text{sech}(t) \, e^{xt} \quad \text{and} \quad \frac{d}{dt} e^{K.(x)t} = R_{HT} e^{K.(x)t} . \]

The sequence of unsigned polynomials has the same lowering op but the raising op
\[ R_T = x + \tan(Dx) \quad \text{and e.g.f.} \quad e^{-it K.(ix)} = \sec(t) \, e^{xt} \]
with a similar differential "heat" or evolution equation.

The Swiss-knife polynomials are also related to the Grassmann polynomials. The raising operator associated with the e.g.f. [A046802] for the polynomials enumerating the cells of the positive Grassmannians is
\[ R_G = x + t + \frac{1 - t}{1 - te^{(1-t)Dx}} \]
\[ = x + t + 1 + tDx + (t + t^2) \frac{D^2}{2!} + (t + 4t^2 + t^3) \frac{D^3}{3!} + \cdots , \]
containing the e.g.f. for the Eulerian polynomials of [A123125]. Then
\[ R^n_G 1 = (G.(0; t) + x)^n = G_n(x; t) \]
are the Grassmann Appell polynomials with the Grassmann polynomials \( G_n(0; t) \) as the base sequence. The first few Grassmann polynomials are

\[ G_0(0; t) = 1 \]
\[ G_1(0; t) = 1 + t = R_G 1|_{x=0} = x + 1 + t|_{x=0} \]
\[ G_2(0; t) = 1 + 3t + t^2 = R_G^2 1|_{x=0} = R_G (G_1(x; t))|_{x=0} \]
\[ G_3(0; t) = 1 + 7t + 7t^2 + t^3 \]
\[ G_4(0; t) = 1 = 15 t + 35 t^2 + 15 t^3 + t^4 . \]

These base polynomials are not an Appell sequence themselves. They serve as a base sequence for the Appell sequence \( G_n(x; t) \), whose e.g.f. satisfies the evolution equation
\[ \frac{d}{dt} e^{t G_n(x; u)} = R_G e^{t G_n(x; u)} . \]

The specialized Appell sequence \( G_n(x; -1) \) has the raising op \( R_{HT} = x - \tanh(D_x) \), so \( G_n(x; -1) = K_n(x) \).

By introducing the Appell sequence relationships, I’m implicitly linking the operators and e.g.f.s to non-crossing partitions and free probability theory free cumulants, moments, convolutions, the Cauchy transform, the Voiculescu polynomials, positroid polytopes, and a whole slew of other combinatorial structures (cf. A134264 and other notes on my website).

See refs in A008292 and below to Hirzebruch, Losev and Manin, et al. for details of connections to enumerative algebraic geometry of the Eulerians (and Euler/tangent numbers), the integer coefficients of the bivariate e.g.f. represented by the general elliptic \( h_E(z) \), whose derivative gives solitons.

The Grassmann raising operator containing the e.g.f. for the Eulerian numbers is directly related to solutions of the KdV equations. Switching from an operator \( R_G(D_v, v) \) to a function \( R(u, v; \alpha) \) while suppressing the subscript and explicitly introducing the \( t \) parameter as \( \alpha \), and a normalized velocity \( \hat{c} = -3c(\alpha) \), let
\[ \rho(u, v) = -\frac{d}{du} R\left( u + \frac{\hat{c}}{3} v, v; \alpha \right) = -\frac{d}{du} \left[ v + \alpha + \frac{1 - \alpha}{1 - \alpha e^{(1-\alpha)(u + \frac{\hat{c}}{3} v)}} \right] \]
\[ = -(1-\alpha)^2 \frac{\alpha \exp\left[ (1-\alpha)(u + \frac{\hat{c}}{3} v) \right]}{\left( 1 - \alpha \exp\left[ (1-\alpha)(u + \frac{\hat{c}}{3} v) \right] \right)^2} = -(1-\alpha)^2 \sum_{n \geq 1} n \alpha^n \exp\left[ n(1-\alpha)(u + \frac{\hat{c}}{3} v) \right] \]
\[ = -[\alpha + (\alpha + \alpha^2)(u + \frac{\hat{c}}{3} v) + (\alpha + 4\alpha^2 + \alpha^3)(u + \frac{\hat{c}}{3} v)^2 + (\alpha + 11\alpha^2 + 11\alpha^3 + \alpha^4)(u + \frac{\hat{c}}{3} v)^3 + \cdots] ; \]
then \( \rho(u, v) \) is a solution of the KdV equation above, i.e.,
\[ \frac{d}{dv} \rho(u, v) - \rho(u, v) \frac{d}{du} \rho(u, v) - \frac{1}{12} \frac{d^3}{du^3} \rho(u, v) = 0 . \]
when, in a suitable region about a point \((u, v)\), \(\hat{c}\) is independent of the coordinates and is
\[
\hat{c} = \frac{D^3_u \rho(u, v) + 6 D_u \rho(u, v)}{4 D_u \rho(u, v)} = \left(\frac{\alpha - 1}{2}\right)^2 = -3c.
\]
for \(\alpha\) non-vanishing and zero for \(\alpha = 0\).

The Grassmann Lie triad with this soliton solution is
\[
Triad_G[h(x, \alpha), h^{-1}(x, \alpha), g_G(x, \alpha)]
\]
with
\[
g_G(x, \alpha) = -\alpha + (1 + \alpha)x - x^2 = (x - \alpha)(1 - x) = \frac{1}{d[h^{-1}(x, \alpha)]/dx} = h'(h^{-1}(x, \alpha), \alpha),
\]
\[
h^{-1}(x, \alpha) = \frac{1}{1 - \alpha \ln \left[\frac{1 - x/\alpha}{1 - x}\right]},
\]
\[
h(u, \alpha) = \alpha e^{\alpha u} - e^u = -[\alpha u + (\alpha + \alpha^2) \frac{u^2}{2!} + (\alpha + 4\alpha^2 + \alpha^3) \frac{u^3}{3!} + (\alpha + 11\alpha^2 + 11\alpha^3 + \alpha^4) \frac{u^4}{4!} + \cdots],
\]
and soliton (with the dependence on \(\alpha\) emphasized)
\[
\rho(u, v, \alpha) = \frac{d}{du} h\left(u + \frac{\hat{c}}{3} v, \alpha\right) = -\alpha + (1 + \alpha) h\left(u + \frac{\hat{c}}{3} v, \alpha\right) - h^2\left(u + \frac{\hat{c}}{3} v, \alpha\right)
\]
with
\[
\hat{c} = \left(\frac{\alpha - 1}{2}\right)^2 = \frac{D^3_u \rho(u, \alpha) + 6 D_u \rho(u, \alpha)}{4 D_u \rho(u, \alpha)} = \frac{1}{2} S_u\{h^{-1}(u, \alpha)\}(g_G(u, \alpha))^2 = -\frac{1}{2} S_u\{h(u, \alpha)\}
\]
\[
= -\frac{1}{2} \left\{D^2_u \ln(\rho(u, v, \alpha)) - \frac{1}{2} [D_u \ln(\rho(u, v, \alpha))]^2\right\} = -3c,
\]
where \(S_u\) is the Schwarzian derivative (quadratic differential) defined below.

For \(\alpha = 1\), the velocity vanishes and \(h(x, 1) = h^{-1}(x, 1) = x/(x - 1)\), the self-inverse Moebius transformation, and \(\rho(x, t, 1) = -1/(x - 1)^2\), satisfying the KdV equation of this section. The raising op then is \(R = x + 1\), giving the Appell sequence \(p_n(x) = (x + 1)^n\) with associated e.g.f. \(e^t e^{tx}\).

(Also note that the normalized \(-\rho(u, v, \alpha)/(\alpha - 1)^2\) has the form of a rotated Koebe function as defined in “The Bieberbach conjecture” by Zorn—let \(u\) and \(v\) be purely imaginary.)
9 Relations of the Appell formalism and Schwarzian to
the viscous Burgers and heat equation ops

From the Appell formalism and the Gesztesy and Holden reference with a change
of notation, the raising operator \( R \) and the Appell e.g.f. \( \psi(u, v) \) and Appell
polynomials \( (p.(x))^n = p_n(x) = (p.(0) + x)^n \) are related by a differential equation
incorporating the differential operator for the viscous Burgers equation on
one side and that for the heat equation on the other. The general Appell e.g.f.
ad op are

\[
\psi(u, v) = \psi(u, 0)e^{uv} = exp[up.(0)]u^{ep} = exp[up.(v)] = e^{uR_1}
\]

and

\[
R(D_v, v) = \frac{d}{du}ln[\psi(u, v)]|_{u=D_v} = v + \frac{d}{dD_v}ln[\psi(D_v, 0)].
\]

Then the two are related by

\[
\frac{d}{dv}(R(u, v)) + R(u, v)\frac{d}{du}R(u, v) + \frac{1}{2}\frac{d^2}{du^2}R(u, v)
\]

\[
= \frac{d}{du}\frac{\psi(u, v)}{\psi(u, v)} + \frac{d}{du}\frac{d^2}{du^2}\psi(u, v),
\]

which, due to the simple dependence on \( v \), reduces to

\[
\frac{d}{du}(R(u, v))^2 + \frac{d^2}{du^2}R(u, v) = \frac{d}{du}\frac{d^2}{du^2}\psi(u, v),
\]

or

\[
(R(u, v))^2 + \frac{d}{du}R(u, v) = \frac{d^2}{du^2}\psi(u, v).
\]

This can be rewritten in terms of the Schwarzian \( S_u\{q(u, v)\} \) with \( \psi(u, v) = D_vq(u, v) \)
as

\[
(R(u, v))^2 + D_v^2ln(\psi(u, v)) = \frac{3}{2}(R(u, v))^2 + S_u\{q(u, v)\} = \frac{D_u^2\psi(u, v)}{\psi(u, v)}.
\]

Rearranging terms, we confirm that this is consistent with the definition of the
Schwarzian

\[
S_u\{q(u, v)\} = D_u^2\psi(u, v) - \frac{3}{2}(R(u, v))^2 = D_u^3q(u, v) - \frac{3}{2}D_uq(u, v)^2 - \frac{3}{2}\left(D_u^2q(u, v)\right)^2.
\]

therefore, the equation incorporating the viscous Burgers differential and the heat equation ops is a morph of the Schwarzian of \( q(u, v) \) in the variable \( u \), the antiderivative of \( \psi(u, v) \).
The Schwarzian defines the potential \( \Omega(u,v) = \frac{1}{2} S_u\{q(u,v)\} \) of a Sturm-Liouville equation with a Hill operator (see Wikipedia and Ovsienko and Tabachnikov ref in the OEIS entry)

\[
\frac{d^2}{du^2} \varphi(u,v) + \Omega(u,v) \varphi(u,v) = 0.
\]

For an excellent overview of the Schwarzian containing many expressions here, see "Old and new on the Schwarzian derivative" by Osgood.

For the Grassmann Lie triad above,

\[
S_u\{h^{-1}(u,\alpha)\} = 2 \left( \frac{\alpha - 1}{2} \right)^2 \frac{1}{[(1-u)(u-\alpha)]^2} = 2e^{-1} \frac{1}{(g_G(u,\alpha))^2}.
\]

Related stuff:

The OEIS and MathOverflow entries noted above have much related material. See also the entries here on the Bernoulli Appells, on the Pascal pyramid and Virasoro algebra, and on the Burgers equation.

"The symmetries of solitons" by Palais (note a \( \text{dlog(det(A(t)))}/\text{dt} \), a gen. Appell sequence?)

"Set partitions and integrable hierarchies" by V. Adler

"Why is there a connection between enumerative geometry and nonlinear waves?" MathOverflow

"Hyperbolic tangent" MathWorld (follow the cross-references to relations to geometry)

"What is the structure of the space of solutions of a nonlinear ODE?" MathOverflow
"Spectral curves, opers, integrable systems" by Ben-Zvi and Frenkel

"Integrable systems: an overview"

"Completely integrable systems, Euclidean Lie algebras, and curves" by M. Adler and Moerbeke

"Singular fiber of the Mumford system and rational solutions to the KdV hierarchy" by Inoue, Vanhaecke, and Yamazaki

"The positive Grassmannian (from a mathematician’s perspective)" by Williams (ocean solitons, KdV and KP eqn., Eulerian numbers appear and Narayana)

"Confluence of hypergeometric functions and integrable hydrodynamic type systems" by Kodama and Konopelchenko (Vandermonde det.)

"The functor of toric varieties associated with Weyl chambers and Losev-Manin moduli spaces" by Batyrev and Blume (pg. 11)

"Eulerian polynomials" by Hirzebruch

"New moduli spaces of pointed curves and pencils of flat connections" by Losev and Manin

"Eulerian polynomials of spherical type" by Cohen

"Manifolds and Modular Forms" by Hirzebruch, Berger, and Jung

"Modular Forms and Topology" by Liu
“Category of vector bundles on algebraic curves and infinite dimensional Grassmannians” by Mulase

“Geometric classification of commutative algebras of ordinary differential operators” by Mulase

“Combinatorial structure of the moduli space of Riemann surfaces and the KP equation” by Mulase and Penkava

“Lectures on the combinatorial structure of the moduli spaces of Riemann surfaces and the Feynman diagram expansion of matrix integrals” by Mulase

“How to understand Grassmannians?” by Baralic

“A gentle introduction to Grassmannians” by Ranganathan

“The Cole-Hopf and Miura transformations revisited” by Gesztesy and Holden

“Krichever maps, Faa di Bruno polynomials, and cohomology in KP theory” by Falqui, Reina, and Zampa

“The Cole-Hopf and Miura transformations revisited” by Gesztesy and Holden

“Geometric realizations of bi-Hamiltonian completely integrable systems” by Beffa

“Bi-Hamilton flows and their geometric representation” by Beffa (slides)

“Moving frames, geometric Poisson brackets, and the KdV-Schwarzian evolution of pure spinors” by Beffa
“The Schwarzian derivative, conformal connections, and Mobius transformations” by Osgood and Stowe

“Differential invariants and invariant differential equations” by Kamran and Olver

“Differential invariants by transvection” by Sanders and Beffa

“Polyakov soldering and second order frames: the role of the Cartan connection” by Lazzarini and Tidei

“Schwarzian integrable systems and the Mobius group: A new connection” by Atkinson

“Second order Lax pairs of nonlinear partial differential equations with Schwarzian forms” by Lou, Tang, Liu, and Fukuyama

“Explanation for the magical powers of the Schwarzian derivative?” MathOverflow

“Some geometric evolution equations arising as geodesic equations on groups of diffeomorphisms including the Hamilton approach” by Michor

“On the geometry of the Virasoro-Bott group” by Michor and Ratiu

“Two dimensional gravity and intersection theory on moduli space” by Witten

“What do water waves have to do with algebraic geometry?” by Peter Miller (slides, also see his lecture notes on KdV)

“Intersection theory, integrable hierarchies, and topological field theory” by Dijkgraaf
“Elie Cartan and geometric Duality” by Bryant

“An introduction to Lie groups and symplectic geometry” by Bryant

“Dirac-Lie systems and Schwarzian equations” by Carinena, Grabowski, Lucas, and Sardon

“Kernvak algebra” by Stevenhagen

“A normal form for elliptic curves” by Edwards

“A survey of the history and properties of solitons” by Bundgaard

“On the origin of the Korteweg-de Vries equation” by Jager

“Integrable viscous conservation laws” by Arsie, Lorenzoni, and Moro