Addendum to Mathemagical Forests

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Sheffer Polynomials for the Generators $z^{m+1}\partial_z$ of the Witt Lie Algebra

In Section 4 C of the paper Mathemagical Forests (MF), the operator expressions

$$\exp(t \ z^{m+1}\partial_z) = \exp(t \ u^m p(z \partial_z)) \big|_{u=z}$$

$$= \exp\left(\left[(1 - m \ t \ u^m)^{-1/m} - 1\right] : z \partial_z :\right) \big|_{u=z} \text{ and}$$

$$p_n(z \partial_z) = z^{-m} n(z^{m+1} \partial_z)^n$$

are presented for the associated functions

$$g(z) = (1 + z)^{m+1},$$

$$h(z) = \partial_z^{-1} (1 / g(z)) = [(1 + z)^{-m} - 1] / (-m) \text{, and}$$

$$h^{-1}(z) = (1 - m z)^{-1/m} - 1.$$

From the formalism of Section 3 of MF, the lowering and raising operators for the Sheffer polynomials $p_n(z)$ are then respectively

$$L = h(\partial_z)$$

$$= [(1 + \partial_z)^{-m} - 1] / (-m)$$

$$= -\frac{1}{m} \sum_{j \geq 1} (-m)^j \partial_z^j = -\frac{1}{m} \sum_{j \geq 1} (-1)^j \binom{m-1+j}{j} \partial_z^j$$
\[ = \sum_{j \geq 1} \ST_{j-1} (-m - 1) \frac{\partial^j}{j!} \]

and

\[ L^{-1} = z g(\partial_z) = z (1 + \partial_z)^{m+1} \]

\[ = z \sum_{j \geq 0} \left( \begin{array}{c} m+1 \\ j \end{array} \right) \partial_z^j = z \left[ 1 + (m + 1) \sum_{j \geq 1} \ST_{j-1} (m) \frac{\partial^j}{j!} \right] \]

where \( \ST_n(x) = \sum_{k \geq 0} \St_1(n,k) x^k = n! \left( \frac{x}{n} \right) \), and \( \St_1(n,k) \) are the Stirling numbers of the first kind.

Consequently,

\[ p_n(z) = e^{-z} p_n(: z \partial_z :) e^z = e^{-z} z^{-m} n (z^{m+1} \partial_z)^n e^z \]

\[ = (L^{-1})^n p_0(z) = [z (1 + \partial_z)^{m+1}]^n p_0(z) = z (1 + \partial_z)^{m+1} p_{n-1}(z) \]

and also from above

\[ \exp[t p_n(z)] = \exp\{[(1 - m t)^{-1/m} - 1] z\} \text{ umbrally.} \]

These expressions are consistent with each other and give

\[ p_0(z) = 1 , \]

\[ p_1(z) = z , \]

\[ p_2(z) = (m + 1) z + z^2 , \]

\[ p_3(z) = (m + 1) (2m + 1) z + 3 (m + 1) z^2 + z^3 , \text{ and} \]

\[ p_4(z) = (m + 1) (2m + 1) (3m + 1) z + (m + 1) (11m + 7) z^2 + 6 (m + 1) z^3 + z^4 . \]

The operator expressions can also be related to powers of the operator \( z \partial_z \) by using the expression

\[ \exp(t \ z \partial_z) = \exp[t \phi(: z \partial_z :)]) = \exp\{[\exp(t) - 1] : z \partial_z :) \}

from Section 4 B of MF, where \( (z \partial_z)^n = \phi_n(: z \partial_z :) \) encodes the Bell / Touchard / exponential polynomials operationally. Then
\[ \exp(t \ z^{m+1} \partial_z) = \exp[t \ u^m p(: z \partial_z :)] \big|_{u=z} \]

\[ = \exp\{ [(1 - m t u^m)^{-1/m} - 1] : z \partial_z : \big|_{u=z} \}
\]

\[ = \exp\{ : z \partial_z : [\exp[\ln[(1 - m t u^m)^{-1/m}]] - 1]} \big|_{u=z} \}
\]

\[ = \exp\{\ln[(1 - m t u^m)^{-1/m}] \phi(: z \partial_z :)] \big|_{u=z} \}
\]

\[ = \exp\{\ln(1 - m t u^m)^{-1/m} z \partial_z \} \big|_{u=z} \]

\[ = (1 - m t u^m)^{-z \partial_z/m} \big|_{u=z} \]

\[ = \sum_{j \geq 0} (-m t z^m)^j \left( \frac{-z \partial_z/m}{j} \right) \]

\[ = \sum_{j \geq 0} \frac{(-m t z^m)^j}{j!} \text{ST}_j \left( -z \partial_z/m \right). \]

This gives for the first few orders

\[ (z^{m+1} \partial_z)^0 = p_0(: z \partial_z :) = 1 \]

\[ (z^{m+1} \partial_z)^1 = z^m p_1(: z \partial_z :) = z^m z \partial_z \]

\[ (z^{m+1} \partial_z)^2 = z^{2m} p_2(: z \partial_z :) = z^{2m} \left( [z \partial_z]^2 + m (z \partial_z)^1 \right) \]

\[ (z^{m+1} \partial_z)^3 = z^{3m} p_3(: z \partial_z :) = z^{3m} \left( [z \partial_z]^3 + 3m (z \partial_z)^2 + 2m^2 (z \partial_z)^1 \right) \]

\[ (z^{m+1} \partial_z)^4 = z^{4m} p_4(: z \partial_z :) = z^{4m} \left( [z \partial_z]^4 + 6m (z \partial_z)^3 + 11m^2 (z \partial_z)^2 + 6m^3 (z \partial_z)^1 \right) \]

\[ = z^{4m} (-m)^4 \text{ST}_4(-\phi(: z \partial_z :)/m) \]

\[ = z^{4m} [\phi_4(: z \partial_z :) + 6m \phi_3(: z \partial_z :) + 11m^2 \phi_2(: z \partial_z :) + 6m^3 \phi_1(: z \partial_z :)] . \]

The coefficients in these expressions, the unsigned Stirling numbers of the first kind in reverse order,

\[ \begin{array}{cccc}
1 & 1 & 1 \\
1 & 3 & 2 \\
1 & 6 & 11 & 6
\end{array} \]

are given in OEIS-A094638.